Multi–Precision Math

Tom St Denis
Algonquin College

Mads Rasmussen
Open Communications Security

Greg Rose
QUALCOMM Australia

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This text has been placed in the public domain. This text corresponds to the v0.39 release of the LibTomMath project.

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Prefaces

When I tell people about my LibTom projects and that I release them as public domain they are often puzzled. They ask why I did it and especially why I continue to work on them for free. The best I can explain it is "Because I can." Which seems odd and perhaps too terse for adult conversation. I often qualify it with "I am able, I am willing." which perhaps explains it better. I am the first to admit there is not anything that special with what I have done. Perhaps others can see that too and then we would have a society to be proud of. My LibTom projects are what I am doing to give back to society in the form of tools and knowledge that can help others in their endeavours.

I started writing this book because it was the most logical task to further my goal of open academia. The LibTomMath source code itself was written to be easy to follow and learn from. There are times, however, where pure C source code does not explain the algorithms properly. Hence this book. The book literally starts with the foundation of the library and works itself outwards to the more complicated algorithms. The use of both pseudo–code and verbatim source code provides a duality of "theory" and "practice" that the computer science students of the world shall appreciate. I never deviate too far from relatively straightforward algebra and I hope that this book can be a valuable learning asset.

This book and indeed much of the LibTom projects would not exist in their current form if it was not for a plethora of kind people donating their time, resources and kind words to help support my work. Writing a text of significant length (along with the source code) is a tiresome and lengthy process. Currently the LibTom project is four years old, comprises of literally thousands of users and over 100,000 lines of source code, TeX and other material. People like Mads and Greg were there at the beginning to encourage me to work well. It is amazing how timely validation from others can boost morale to continue the project. Definitely my parents were there for me by providing room and board

during the many months of work in 2003.

To my many friends whom I have met through the years I thank you for the good times and the words of encouragement. I hope I honour your kind gestures with this project.

Open Source. Open Academia. Open Minds.

Tom St Denis

I found the opportunity to work with Tom appealing for several reasons, not only could I broaden my own horizons, but also contribute to educate others facing the problem of having to handle big number mathematical calculations.

This book is Tom's child and he has been caring and fostering the project ever since the beginning with a clear mind of how he wanted the project to turn out. I have helped by proofreading the text and we have had several discussions about the layout and language used.

I hold a masters degree in cryptography from the University of Southern Denmark and have always been interested in the practical aspects of cryptography.

Having worked in the security consultancy business for several years in São Paulo, Brazil, I have been in touch with a great deal of work in which multiple precision mathematics was needed. Understanding the possibilities for speeding up multiple precision calculations is often very important since we deal with outdated machine architecture where modular reductions, for example, become painfully slow.

This text is for people who stop and wonder when first examining algorithms such as RSA for the first time and asks themselves, "You tell me this is only secure for large numbers, fine; but how do you implement these numbers?"

Mads Rasmussen
São Paulo - SP
Brazil

It's all because I broke my leg. That just happened to be at about the same time that Tom asked for someone to review the section of the book about Karatsuba multiplication. I was laid up, alone and immobile, and thought "Why not?" I vaguely knew what Karatsuba multiplication was, but not really, so I thought I could help, learn, and stop myself from watching daytime cable TV, all at once.

At the time of writing this, I've still not met Tom or Mads in meatspace. I've been following Tom's progress since his first splash on the sci.crypt Usenet news group. I watched him go from a clueless newbie, to the cryptographic equivalent of a reformed smoker, to a real contributor to the field, over a period of about two years. I've been impressed with his obvious intelligence, and astounded by his productivity. Of course, he's young enough to be my own child, so he doesn't have my problems with staying awake.

When I reviewed that single section of the book, in its very earliest form, I was very pleasantly surprised. So I decided to collaborate more fully, and at least review all of it, and perhaps write some bits too. There's still a long way to go with it, and I have watched a number of close friends go through the mill of publication, so I think that the way to go is longer than Tom thinks it is. Nevertheless, it's a good effort, and I'm pleased to be involved with it.

Greg Rose, Sydney, Australia, June 2003.

Chapter 1

Introduction

1.1 Multiple Precision Arithmetic

1.1.1 What is Multiple Precision Arithmetic?

When we think of long-hand arithmetic such as addition or multiplication we rarely consider the fact that we instinctively raise or lower the precision of the numbers we are dealing with. For example, in decimal we almost immediate can reason that 7 times 6 is 42. However, 42 has two digits of precision as opposed to one digit we started with. Further multiplications of say 3 result in a larger precision result 126. In these few examples we have multiple precisions for the numbers we are working with. Despite the various levels of precision a single subset[1] of algorithms can be designed to accomodate them.

By way of comparison a fixed or single precision operation would lose precision on various operations. For example, in the decimal system with fixed precision $6 \cdot 7 = 2$.

Essentially at the heart of computer based multiple precision arithmetic are the same long-hand algorithms taught in schools to manually add, subtract, multiply and divide.

1.1.2 The Need for Multiple Precision Arithmetic

The most prevalent need for multiple precision arithmetic, often referred to as "bignum" math, is within the implementation of public-key cryptography

[1] With the occasional optimization.

algorithms. Algorithms such as RSA [11] and Diffie-Hellman [12] require integers of significant magnitude to resist known cryptanalytic attacks. For example, at the time of this writing a typical RSA modulus would be at least greater than 10^{309}. However, modern programming languages such as ISO C [18] and Java [19] only provide instrinsic support for integers which are relatively small and single precision.

Data Type	**Range**
char	$-128 \ldots 127$
short	$-32768 \ldots 32767$
long	$-2147483648 \ldots 2147483647$
long long	$-9223372036854775808 \ldots 9223372036854775807$

Figure 1.1: Typical Data Types for the C Programming Language

The largest data type guaranteed to be provided by the ISO C programming language[2] can only represent values up to 10^{19} as shown in figure 1.1. On its own the C language is insufficient to accomodate the magnitude required for the problem at hand. An RSA modulus of magnitude 10^{19} could be trivially factored[3] on the average desktop computer, rendering any protocol based on the algorithm insecure. Multiple precision algorithms solve this very problem by extending the range of representable integers while using single precision data types.

Most advancements in fast multiple precision arithmetic stem from the need for faster and more efficient cryptographic primitives. Faster modular reduction and exponentiation algorithms such as Barrett's algorithm, which have appeared in various cryptographic journals, can render algorithms such as RSA and Diffie-Hellman more efficient. In fact, several major companies such as RSA Security, Certicom and Entrust have built entire product lines on the implementation and deployment of efficient algorithms.

However, cryptography is not the only field of study that can benefit from fast multiple precision integer routines. Another auxiliary use of multiple precision integers is high precision floating point data types. The basic IEEE [13] standard floating point type is made up of an integer mantissa q, an exponent e and a sign bit s. Numbers are given in the form $n = q \cdot b^e \cdot -1^s$ where $b = 2$

[2] As per the ISO C standard. However, each compiler vendor is allowed to augment the precision as they see fit.

[3] A Pollard-Rho factoring would take only 2^{16} time.

is the most common base for IEEE. Since IEEE floating point is meant to be implemented in hardware the precision of the mantissa is often fairly small (*23, 48 and 64 bits*). The mantissa is merely an integer and a multiple precision integer could be used to create a mantissa of much larger precision than hardware alone can efficiently support. This approach could be useful where scientific applications must minimize the total output error over long calculations.

Yet another use for large integers is within arithmetic on polynomials of large characteristic (i.e. $GF(p)[x]$ for large p). In fact the library discussed within this text has already been used to form a polynomial basis library[4].

1.1.3 Benefits of Multiple Precision Arithmetic

The benefit of multiple precision representations over single or fixed precision representations is that no precision is lost while representing the result of an operation which requires excess precision. For example, the product of two n-bit integers requires at least $2n$ bits of precision to be represented faithfully. A multiple precision algorithm would augment the precision of the destination to accomodate the result while a single precision system would truncate excess bits to maintain a fixed level of precision.

It is possible to implement algorithms which require large integers with fixed precision algorithms. For example, elliptic curve cryptography (ECC) is often implemented on smartcards by fixing the precision of the integers to the maximum size the system will ever need. Such an approach can lead to vastly simpler algorithms which can accomodate the integers required even if the host platform cannot natively accomodate them[5]. However, as efficient as such an approach may be, the resulting source code is not normally very flexible. It cannot, at runtime, accomodate inputs of higher magnitude than the designer anticipated.

Multiple precision algorithms have the most overhead of any style of arithmetic. For the the most part the overhead can be kept to a minimum with careful planning, but overall, it is not well suited for most memory starved platforms. However, multiple precision algorithms do offer the most flexibility in terms of the magnitude of the inputs. That is, the same algorithms based on multiple precision integers can accomodate any reasonable size input without the designer's explicit forethought. This leads to lower cost of ownership for the code as it only has to be written and tested once.

[4] See `http://poly.libtomcrypt.org` for more details.

[5] For example, the average smartcard processor has an 8 bit accumulator.

1.2 Purpose of This Text

The purpose of this text is to instruct the reader regarding how to implement efficient multiple precision algorithms. That is to not only explain a limited subset of the core theory behind the algorithms but also the various "house keeping" elements that are neglected by authors of other texts on the subject. Several well reknowned texts [1, 2] give considerably detailed explanations of the theoretical aspects of algorithms and often very little information regarding the practical implementation aspects.

In most cases how an algorithm is explained and how it is actually implemented are two very different concepts. For example, the Handbook of Applied Cryptography (*HAC*), algorithm 14.7 on page 594, gives a relatively simple algorithm for performing multiple precision integer addition. However, the description lacks any discussion concerning the fact that the two integer inputs may be of differing magnitudes. As a result the implementation is not as simple as the text would lead people to believe. Similarly the division routine (*algorithm 14.20, pp. 598*) does not discuss how to handle sign or handle the dividend's decreasing magnitude in the main loop (*step #3*).

Both texts also do not discuss several key optimal algorithms required such as "Comba" and Karatsuba multipliers and fast modular inversion, which we consider practical oversights. These optimal algorithms are vital to achieve any form of useful performance in non-trivial applications.

To solve this problem the focus of this text is on the practical aspects of implementing a multiple precision integer package. As a case study the "LibTomMath"[6] package is used to demonstrate algorithms with real implementations[7] that have been field tested and work very well. The LibTomMath library is freely available on the Internet for all uses and this text discusses a very large portion of the inner workings of the library.

The algorithms that are presented will always include at least one "pseudo-code" description followed by the actual C source code that implements the algorithm. The pseudo-code can be used to implement the same algorithm in other programming languages as the reader sees fit.

This text shall also serve as a walkthrough of the creation of multiple precision algorithms from scratch. Showing the reader how the algorithms fit together as well as where to start on various taskings.

[6] Available at `http://math.libtomcrypt.com`

[7] In the ISO C programming language.

1.3 Discussion and Notation

1.3.1 Notation

A multiple precision integer of n-digits shall be denoted as $x = (x_{n-1}, \ldots, x_1, x_0)_\beta$ and represent the integer $x \equiv \sum_{i=0}^{n-1} x_i \beta^i$. The elements of the array x are said to be the radix β digits of the integer. For example, $x = (1, 2, 3)_{10}$ would represent the integer $1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 = 123$.

The term "mp_int" shall refer to a composite structure which contains the digits of the integer it represents, as well as auxilary data required to manipulate the data. These additional members are discussed further in section 2.2.1. For the purposes of this text a "multiple precision integer" and an "mp_int" are assumed to be synonymous. When an algorithm is specified to accept an mp_int variable it is assumed the various auxliary data members are present as well. An expression of the type *variablename.item* implies that it should evaluate to the member named "item" of the variable. For example, a string of characters may have a member "length" which would evaluate to the number of characters in the string. If the string a equals "hello" then it follows that $a.length = 5$.

For certain discussions more generic algorithms are presented to help the reader understand the final algorithm used to solve a given problem. When an algorithm is described as accepting an integer input it is assumed the input is a plain integer with no additional multiple-precision members. That is, algorithms that use integers as opposed to mp_ints as inputs do not concern themselves with the housekeeping operations required such as memory management. These algorithms will be used to establish the relevant theory which will subsequently be used to describe a multiple precision algorithm to solve the same problem.

1.3.2 Precision Notation

The variable β represents the radix of a single digit of a multiple precision integer and must be of the form q^p for $q, p \in \mathbb{Z}^+$. A single precision variable must be able to represent integers in the range $0 \le x < q\beta$ while a double precision variable must be able to represent integers in the range $0 \le x < q\beta^2$. The extra radix-q factor allows additions and subtractions to proceed without truncation of the carry. Since all modern computers are binary, it is assumed that q is two.

Within the source code that will be presented for each algorithm, the data type **mp_digit** will represent a single precision integer type, while, the data type **mp_word** will represent a double precision integer type. In several algorithms

(notably the Comba routines) temporary results will be stored in arrays of double precision mp_words. For the purposes of this text x_j will refer to the j'th digit of a single precision array and $\hat{x}_j$ will refer to the j'th digit of a double precision array. Whenever an expression is to be assigned to a double precision variable it is assumed that all single precision variables are promoted to double precision during the evaluation. Expressions that are assigned to a single precision variable are truncated to fit within the precision of a single precision data type.

For example, if $\beta = 10^2$ a single precision data type may represent a value in the range $0 \le x < 10^3$, while a double precision data type may represent a value in the range $0 \le x < 10^5$. Let $a = 23$ and $b = 49$ represent two single precision variables. The single precision product shall be written as $c \leftarrow a \cdot b$ while the double precision product shall be written as $\hat{c} \leftarrow a \cdot b$. In this particular case, $\hat{c} = 1127$ and $c = 127$. The most significant digit of the product would not fit in a single precision data type and as a result $c \neq \hat{c}$.

1.3.3 Algorithm Inputs and Outputs

Within the algorithm descriptions all variables are assumed to be scalars of either single or double precision as indicated. The only exception to this rule is when variables have been indicated to be of type mp_int. This distinction is important as scalars are often used as array indicies and various other counters.

1.3.4 Mathematical Expressions

The $\lfloor$ $\rfloor$ brackets imply an expression truncated to an integer not greater than the expression itself. For example, $\lfloor 5.7 \rfloor = 5$. Similarly the $\lceil$ $\rceil$ brackets imply an expression rounded to an integer not less than the expression itself. For example, $\lceil 5.1 \rceil = 6$. Typically when the / division symbol is used the intention is to perform an integer division with truncation. For example, $5/2 = 2$ which will often be written as $\lfloor 5/2 \rfloor = 2$ for clarity. When an expression is written as a fraction a real value division is implied, for example $\frac{5}{2} = 2.5$.

The norm of a multiple precision integer, for example $||x||$, will be used to represent the number of digits in the representation of the integer. For example, $||123|| = 3$ and $||79452|| = 5$.

1.3.5 Work Effort

To measure the efficiency of the specified algorithms, a modified big-Oh notation is used. In this system all single precision operations are considered to have the same cost[8]. That is a single precision addition, multiplication and division are assumed to take the same time to complete. While this is generally not true in practice, it will simplify the discussions considerably.

Some algorithms have slight advantages over others which is why some constants will not be removed in the notation. For example, a normal baseline multiplication (section 5.2.1) requires $O(n^2)$ work while a baseline squaring (section 5.3) requires $O(\frac{n^2+n}{2})$ work. In standard big-Oh notation these would both be said to be equivalent to $O(n^2)$. However, in the context of the this text this is not the case as the magnitude of the inputs will typically be rather small. As a result small constant factors in the work effort will make an observable difference in algorithm efficiency.

All of the algorithms presented in this text have a polynomial time work level. That is, of the form $O(n^k)$ for $n, k \in \mathbb{Z}^+$. This will help make useful comparisons in terms of the speed of the algorithms and how various optimizations will help pay off in the long run.

1.4 Exercises

Within the more advanced chapters a section will be set aside to give the reader some challenging exercises related to the discussion at hand. These exercises are not designed to be prize winning problems, but instead to be thought provoking. Wherever possible the problems are forward minded, stating problems that will be answered in subsequent chapters. The reader is encouraged to finish the exercises as they appear to get a better understanding of the subject material.

That being said, the problems are designed to affirm knowledge of a particular subject matter. Students in particular are encouraged to verify they can answer the problems correctly before moving on.

Similar to the exercises of [1, pp. ix] these exercises are given a scoring system based on the difficulty of the problem. However, unlike [1] the problems do not get nearly as hard. The scoring of these exercises ranges from one (the easiest) to five (the hardest). The following table sumarizes the scoring system used.

[8] Except where explicitly noted.

[1]	An easy problem that should only take the reader a manner of minutes to solve. Usually does not involve much computer time to solve.
[2]	An easy problem that involves a marginal amount of computer time usage. Usually requires a program to be written to solve the problem.
[3]	A moderately hard problem that requires a non-trivial amount of work. Usually involves trivial research and development of new theory from the perspective of a student.
[4]	A moderately hard problem that involves a non-trivial amount of work and research, the solution to which will demonstrate a higher mastery of the subject matter.
[5]	A hard problem that involves concepts that are difficult for a novice to solve. Solutions to these problems will demonstrate a complete mastery of the given subject.

Figure 1.2: Exercise Scoring System

Problems at the first level are meant to be simple questions that the reader can answer quickly without programming a solution or devising new theory. These problems are quick tests to see if the material is understood. Problems at the second level are also designed to be easy but will require a program or algorithm to be implemented to arrive at the answer. These two levels are essentially entry level questions.

Problems at the third level are meant to be a bit more difficult than the first two levels. The answer is often fairly obvious but arriving at an exacting solution requires some thought and skill. These problems will almost always involve devising a new algorithm or implementing a variation of another algorithm previously presented. Readers who can answer these questions will feel comfortable with the concepts behind the topic at hand.

Problems at the fourth level are meant to be similar to those of the level three questions except they will require additional research to be completed. The reader will most likely not know the answer right away, nor will the text provide the exact details of the answer until a subsequent chapter.

Problems at the fifth level are meant to be the hardest problems relative to all the other problems in the chapter. People who can correctly answer fifth level problems have a mastery of the subject matter at hand.

Often problems will be tied together. The purpose of this is to start a chain

of thought that will be discussed in future chapters. The reader is encouraged to answer the follow-up problems and try to draw the relevance of problems.

1.5 Introduction to LibTomMath

1.5.1 What is LibTomMath?

LibTomMath is a free and open source multiple precision integer library written entirely in portable ISO C. By portable it is meant that the library does not contain any code that is computer platform dependent or otherwise problematic to use on any given platform.

The library has been successfully tested under numerous operating systems including Unix[9], MacOS, Windows, Linux, PalmOS and on standalone hardware such as the Gameboy Advance. The library is designed to contain enough functionality to be able to develop applications such as public key cryptosystems and still maintain a relatively small footprint.

1.5.2 Goals of LibTomMath

Libraries which obtain the most efficiency are rarely written in a high level programming language such as C. However, even though this library is written entirely in ISO C, considerable care has been taken to optimize the algorithm implementations within the library. Specifically the code has been written to work well with the GNU C Compiler (*GCC*) on both x86 and ARM processors. Wherever possible, highly efficient algorithms, such as Karatsuba multiplication, sliding window exponentiation and Montgomery reduction have been provided to make the library more efficient.

Even with the nearly optimal and specialized algorithms that have been included the Application Programing Interface (*API*) has been kept as simple as possible. Often generic place holder routines will make use of specialized algorithms automatically without the developer's specific attention. One such example is the generic multiplication algorithm **mp_mul()** which will automatically use Toom–Cook, Karatsuba, Comba or baseline multiplication based on the magnitude of the inputs and the configuration of the library.

Making LibTomMath as efficient as possible is not the only goal of the LibTomMath project. Ideally the library should be source compatible with another popular library which makes it more attractive for developers to use.

[9] All of these trademarks belong to their respective rightful owners.

In this case the MPI library was used as a API template for all the basic functions. MPI was chosen because it is another library that fits in the same niche as LibTomMath. Even though LibTomMath uses MPI as the template for the function names and argument passing conventions, it has been written from scratch by Tom St Denis.

The project is also meant to act as a learning tool for students, the logic being that no easy-to-follow "bignum" library exists which can be used to teach computer science students how to perform fast and reliable multiple precision integer arithmetic. To this end the source code has been given quite a few comments and algorithm discussion points.

1.6 Choice of LibTomMath

LibTomMath was chosen as the case study of this text not only because the author of both projects is one and the same but for more worthy reasons. Other libraries such as GMP [14], MPI [15], LIP [17] and OpenSSL [16] have multiple precision integer arithmetic routines but would not be ideal for this text for reasons that will be explained in the following sub-sections.

1.6.1 Code Base

The LibTomMath code base is all portable ISO C source code. This means that there are no platform dependent conditional segments of code littered throughout the source. This clean and uncluttered approach to the library means that a developer can more readily discern the true intent of a given section of source code without trying to keep track of what conditional code will be used.

The code base of LibTomMath is well organized. Each function is in its own separate source code file which allows the reader to find a given function very quickly. On average there are 76 lines of code per source file which makes the source very easily to follow. By comparison MPI and LIP are single file projects making code tracing very hard. GMP has many conditional code segments which also hinder tracing.

When compiled with GCC for the x86 processor and optimized for speed the entire library is approximately 100KiB[10] which is fairly small compared to GMP (over 250KiB). LibTomMath is slightly larger than MPI (which compiles to about 50KiB) but LibTomMath is also much faster and more complete than MPI.

[10]The notation "KiB" means 2^{10} octets, similarly "MiB" means 2^{20} octets.

1.6.2 API Simplicity

LibTomMath is designed after the MPI library and shares the API design. Quite often programs that use MPI will build with LibTomMath without change. The function names correlate directly to the action they perform. Almost all of the functions share the same parameter passing convention. The learning curve is fairly shallow with the API provided which is an extremely valuable benefit for the student and developer alike.

The LIP library is an example of a library with an API that is awkward to work with. LIP uses function names that are often "compressed" to illegible short hand. LibTomMath does not share this characteristic.

The GMP library also does not return error codes. Instead it uses a POSIX.1 [23] signal system where errors are signaled to the host application. This happens to be the fastest approach but definitely not the most versatile. In effect a math error (i.e. invalid input, heap error, etc) can cause a program to stop functioning which is definitely undersireable in many situations.

1.6.3 Optimizations

While LibTomMath is certainly not the fastest library (GMP often beats LibTomMath by a factor of two) it does feature a set of optimal algorithms for tasks such as modular reduction, exponentiation, multiplication and squaring. GMP and LIP also feature such optimizations while MPI only uses baseline algorithms with no optimizations. GMP lacks a few of the additional modular reduction optimizations that LibTomMath features[11].

LibTomMath is almost always an order of magnitude faster than the MPI library at computationally expensive tasks such as modular exponentiation. In the grand scheme of "bignum" libraries LibTomMath is faster than the average library and usually slower than the best libraries such as GMP and OpenSSL by only a small factor.

1.6.4 Portability and Stability

LibTomMath will build "out of the box" on any platform equipped with a modern version of the GNU C Compiler (*GCC*). This means that without changes the library will build without configuration or setting up any variables. LIP and MPI will build "out of the box" as well but have numerous known bugs. Most

[11] At the time of this writing GMP only had Barrett and Montgomery modular reduction algorithms.

notably the author of MPI has recently stopped working on his library and LIP has long since been discontinued.

GMP requires a configuration script to run and will not build out of the box. GMP and LibTomMath are still in active development and are very stable across a variety of platforms.

1.6.5 Choice

LibTomMath is a relatively compact, well documented, highly optimized and portable library which seems only natural for the case study of this text. Various source files from the LibTomMath project will be included within the text. However, the reader is encouraged to download their own copy of the library to actually be able to work with the library.

Chapter 2

Getting Started

2.1 Library Basics

The trick to writing any useful library of source code is to build a solid foundation and work outwards from it. First, a problem along with allowable solution parameters should be identified and analyzed. In this particular case the inability to accomodate multiple precision integers is the problem. Futhermore, the solution must be written as portable source code that is reasonably efficient across several different computer platforms.

After a foundation is formed the remainder of the library can be designed and implemented in a hierarchical fashion. That is, to implement the lowest level dependencies first and work towards the most abstract functions last. For example, before implementing a modular exponentiation algorithm one would implement a modular reduction algorithm. By building outwards from a base foundation instead of using a parallel design methodology the resulting project is highly modular. Being highly modular is a desirable property of any project as it often means the resulting product has a small footprint and updates are easy to perform.

Usually when I start a project I will begin with the header files. I define the data types I think I will need and prototype the initial functions that are not dependent on other functions (within the library). After I implement these base functions I prototype more dependent functions and implement them. The process repeats until I implement all of the functions I require. For example, in the case of LibTomMath I implemented functions such as mp_init() well before

I implemented mp_mul() and even further before I implemented mp_exptmod(). As an example as to why this design works note that the Karatsuba and Toom-Cook multipliers were written *after* the dependent function mp_exptmod() was written. Adding the new multiplication algorithms did not require changes to the mp_exptmod() function itself and lowered the total cost of ownership (*so to speak*) and of development for new algorithms. This methodology allows new algorithms to be tested in a complete framework with relative ease.

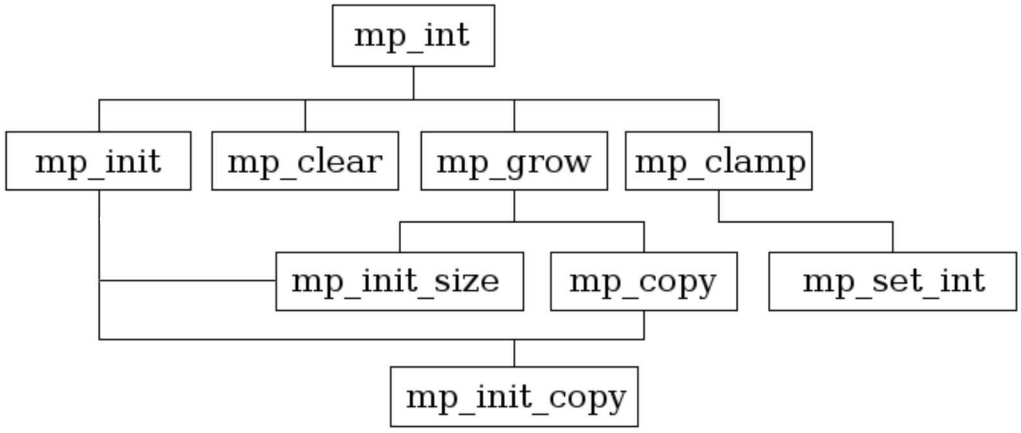


Figure 2.1: Design Flow of the First Few Original LibTomMath Functions.

Only after the majority of the functions were in place did I pursue a less hierarchical approach to auditing and optimizing the source code. For example, one day I may audit the multipliers and the next day the polynomial basis functions.

It only makes sense to begin the text with the preliminary data types and support algorithms required as well. This chapter discusses the core algorithms of the library which are the dependents for every other algorithm.

2.2 What is a Multiple Precision Integer?

Recall that most programming languages, in particular ISO C [18], only have fixed precision data types that on their own cannot be used to represent values

larger than their precision will allow. The purpose of multiple precision algorithms is to use fixed precision data types to create and manipulate multiple precision integers which may represent values that are very large.

As a well known analogy, school children are taught how to form numbers larger than nine by prepending more radix ten digits. In the decimal system the largest single digit value is 9. However, by concatenating digits together larger numbers may be represented. Newly prepended digits (*to the left*) are said to be in a different power of ten column. That is, the number 123 can be described as having a 1 in the hundreds column, 2 in the tens column and 3 in the ones column. Or more formally $123 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0$. Computer based multiple precision arithmetic is essentially the same concept. Larger integers are represented by adjoining fixed precision computer words with the exception that a different radix is used.

What most people probably do not think about explicitly are the various other attributes that describe a multiple precision integer. For example, the integer 154_{10} has two immediately obvious properties. First, the integer is positive, that is the sign of this particular integer is positive as opposed to negative. Second, the integer has three digits in its representation. There is an additional property that the integer posesses that does not concern pencil-and-paper arithmetic. The third property is how many digits placeholders are available to hold the integer.

The human analogy of this third property is ensuring there is enough space on the paper to write the integer. For example, if one starts writing a large number too far to the right on a piece of paper they will have to erase it and move left. Similarly, computer algorithms must maintain strict control over memory usage to ensure that the digits of an integer will not exceed the allowed boundaries. These three properties make up what is known as a multiple precision integer or mp_int for short.

2.2.1 The mp_int Structure

The mp_int structure is the ISO C based manifestation of what represents a multiple precision integer. The ISO C standard does not provide for any such data type but it does provide for making composite data types known as structures. The following is the structure definition used within LibTomMath.

The mp_int structure (fig. 2.2) can be broken down as follows.

1. The **used** parameter denotes how many digits of the array **dp** contain the digits used to represent a given integer. The **used** count must be positive

```
typedef struct {
   int used, alloc, sign;
   mp_digit *dp;
} mp_int;
```

Figure 2.2: The mp_int Structure

(or zero) and may not exceed the **alloc** count.

2. The **alloc** parameter denotes how many digits are available in the array to use by functions before it has to increase in size. When the **used** count of a result would exceed the **alloc** count all of the algorithms will automatically increase the size of the array to accommodate the precision of the result.

3. The pointer **dp** points to a dynamically allocated array of digits that represent the given multiple precision integer. It is padded with (**alloc** − **used**) zero digits. The array is maintained in a least significant digit order. As a pencil and paper analogy the array is organized such that the right most digits are stored first starting at the location indexed by zero[1] in the array. For example, if **dp** contains $\{a, b, c, \ldots\}$ where $\mathbf{dp}_0 = a$, $\mathbf{dp}_1 = b$, $\mathbf{dp}_2 = c$, ... then it would represent the integer $a + b\beta + c\beta^2 + \ldots$

4. The **sign** parameter denotes the sign as either zero/positive (**MP_ZPOS**) or negative (**MP_NEG**).

Valid mp_int Structures

Several rules are placed on the state of an mp_int structure and are assumed to be followed for reasons of efficiency. The only exceptions are when the structure is passed to initialization functions such as mp_init() and mp_init_copy().

1. The value of **alloc** may not be less than one. That is **dp** always points to a previously allocated array of digits.

2. The value of **used** may not exceed **alloc** and must be greater than or equal to zero.

[1] In C all arrays begin at zero.

3. The value of **used** implies the digit at index $(used - 1)$ of the **dp** array is non-zero. That is, leading zero digits in the most significant positions must be trimmed.

   (a) Digits in the **dp** array at and above the **used** location must be zero.

4. The value of **sign** must be **MP_ZPOS** if **used** is zero; this represents the mp_int value of zero.

2.3 Argument Passing

A convention of argument passing must be adopted early on in the development of any library. Making the function prototypes consistent will help eliminate many headaches in the future as the library grows to significant complexity. In LibTomMath the multiple precision integer functions accept parameters from left to right as pointers to mp_int structures. That means that the source (input) operands are placed on the left and the destination (output) on the right. Consider the following examples.

```
mp_mul(&a, &b, &c);   /* c = a * b */
mp_add(&a, &b, &a);   /* a = a + b */
mp_sqr(&a, &b);       /* b = a * a */
```

The left to right order is a fairly natural way to implement the functions since it lets the developer read aloud the functions and make sense of them. For example, the first function would read "multiply a and b and store in c".

Certain libraries (*LIP by Lenstra for instance*) accept parameters the other way around, to mimic the order of assignment expressions. That is, the destination (output) is on the left and arguments (inputs) are on the right. In truth, it is entirely a matter of preference. In the case of LibTomMath the convention from the MPI library has been adopted.

Another very useful design consideration, provided for in LibTomMath, is whether to allow argument sources to also be a destination. For example, the second example (*mp_add*) adds a to b and stores in a. This is an important feature to implement since it allows the calling functions to cut down on the number of variables it must maintain. However, to implement this feature specific care has to be given to ensure the destination is not modified before the source is fully read.

2.4 Return Values

A well implemented application, no matter what its purpose, should trap as many runtime errors as possible and return them to the caller. By catching runtime errors a library can be guaranteed to prevent undefined behaviour. However, the end developer can still manage to cause a library to crash. For example, by passing an invalid pointer an application may fault by dereferencing memory not owned by the application.

In the case of LibTomMath the only errors that are checked for are related to inappropriate inputs (division by zero for instance) and memory allocation errors. It will not check that the mp_int passed to any function is valid nor will it check pointers for validity. Any function that can cause a runtime error will return an error code as an **int** data type with one of the following values (fig 2.3).

Value	Meaning
MP_OKAY	The function was successful
MP_VAL	One of the input value(s) was invalid
MP_MEM	The function ran out of heap memory

Figure 2.3: LibTomMath Error Codes

When an error is detected within a function it should free any memory it allocated, often during the initialization of temporary mp_ints, and return as soon as possible. The goal is to leave the system in the same state it was when the function was called. Error checking with this style of API is fairly simple.

```
int err;
if ((err = mp_add(&a, &b, &c)) != MP_OKAY) {
   printf("Error: %s\n", mp_error_to_string(err));
   exit(EXIT_FAILURE);
}
```

The GMP [14] library uses C style *signals* to flag errors which is of questionable use. Not all errors are fatal and it was not deemed ideal by the author of LibTomMath to force developers to have signal handlers for such cases.

2.5 Initialization and Clearing

The logical starting point when actually writing multiple precision integer functions is the initialization and clearing of the mp_int structures. These two algorithms will be used by the majority of the higher level algorithms.

Given the basic mp_int structure an initialization routine must first allocate memory to hold the digits of the integer. Often it is optimal to allocate a sufficiently large pre-set number of digits even though the initial integer will represent zero. If only a single digit were allocated quite a few subsequent reallocations would occur when operations are performed on the integers. There is a tradeoff between how many default digits to allocate and how many reallocations are tolerable. Obviously allocating an excessive amount of digits initially will waste memory and become unmanageable.

If the memory for the digits has been successfully allocated then the rest of the members of the structure must be initialized. Since the initial state of an mp_int is to represent the zero integer, the allocated digits must be set to zero. The **used** count set to zero and **sign** set to **MP_ZPOS**.

2.5.1 Initializing an mp_int

An mp_int is said to be initialized if it is set to a valid, preferably default, state such that all of the members of the structure are set to valid values. The mp_init algorithm will perform such an action.

Algorithm **mp_init**.
Input. An mp_int a
Output. Allocate memory and initialize a to a known valid mp_int state.

1. Allocate memory for **MP_PREC** digits.
2. If the allocation failed return(MP_MEM)
3. for n from 0 to $MP_PREC - 1$ do
   3.1 $a_n \leftarrow 0$
4. $a.sign \leftarrow MP_ZPOS$
5. $a.used \leftarrow 0$
6. $a.alloc \leftarrow MP_PREC$
7. Return(MP_OKAY)

Figure 2.4: Algorithm mp_init

Algorithm mp_init. The purpose of this function is to initialize an mp_int structure so that the rest of the library can properly manipulte it. It is assumed that the input may not have had any of its members previously initialized which is certainly a valid assumption if the input resides on the stack.

Before any of the members such as **sign**, **used** or **alloc** are initialized the memory for the digits is allocated. If this fails the function returns before setting any of the other members. The **MP_PREC** name represents a constant[2] used to dictate the minimum precision of newly initialized mp_int integers. Ideally, it is at least equal to the smallest precision number you'll be working with.

Allocating a block of digits at first instead of a single digit has the benefit of lowering the number of usually slow heap operations later functions will have to perform in the future. If **MP_PREC** is set correctly the slack memory and the number of heap operations will be trivial.

Once the allocation has been made the digits have to be set to zero as well as the **used**, **sign** and **alloc** members initialized. This ensures that the mp_int will always represent the default state of zero regardless of the original condition of the input.

Remark. This function introduces the idiosyncrasy that all iterative loops, commonly initiated with the "for" keyword, iterate incrementally when the "to" keyword is placed between two expressions. For example, "for a from b to c do" means that a subsequent expression (or body of expressions) are to be evaluated upto $c - b$ times so long as $b \le c$. In each iteration the variable a is substituted for a new integer that lies inclusively between b and c. If $b > c$ occured the loop would not iterate. By contrast if the "downto" keyword were used in place of "to" the loop would iterate decrementally.

File: bn_mp_init.c

```
001   #ifdef BN_MP_INIT_C
...
```

One immediate observation of this initializtion function is that it does not return a pointer to a mp_int structure. It is assumed that the caller has already allocated memory for the mp_int structure, typically on the application stack. The call to mp_init() is used only to initialize the members of the structure to a known default state.

Here we see (line 21) the memory allocation is performed first. This allows us to exit cleanly and quickly if there is an error. If the allocation fails the routine will return **MP_MEM** to the caller to indicate there was a memory error.

[2]Defined in the "tommath.h" header file within LibTomMath.

The function XMALLOC is what actually allocates the memory. Technically XMALLOC is not a function but a macro defined in "tommath.h". By default, XMALLOC will evaluate to malloc() which is the C library's built–in memory allocation routine.

In order to assure the mp_int is in a known state the digits must be set to zero. On most platforms this could have been accomplished by using calloc() instead of malloc(). However, to correctly initialize a integer type to a given value in a portable fashion you have to actually assign the value. The for loop (line 27) performs this required operation.

After the memory has been successfully initialized the remainder of the members are initialized (lines 31 through 32) to their respective default states. At this point the algorithm has succeeded and a success code is returned to the calling function. If this function returns **MP_OKAY** it is safe to assume the mp_int structure has been properly initialized and is safe to use with other functions within the library.

2.5.2 Clearing an mp_int

When an mp_int is no longer required by the application, the memory that has been allocated for its digits must be returned to the application's memory pool with the mp_clear algorithm.

Algorithm **mp_clear**.
Input. An mp_int a
Output. The memory for a shall be deallocated.

1. If a has been previously freed then return(MP_OKAY).
2. for n from 0 to $a.used - 1$ do
   2.1 $a_n \leftarrow 0$
3. Free the memory allocated for the digits of a.
4. $a.used \leftarrow 0$
5. $a.alloc \leftarrow 0$
6. $a.sign \leftarrow MP_ZPOS$
7. Return(MP_OKAY).

Figure 2.5: Algorithm mp_clear

Algorithm mp_clear. This algorithm accomplishes two goals. First, it

clears the digits and the other mp_int members. This ensures that if a developer accidentally re-uses a cleared structure it is less likely to cause problems. The second goal is to free the allocated memory.

The logic behind the algorithm is extended by marking cleared mp_int structures so that subsequent calls to this algorithm will not try to free the memory multiple times. Cleared mp_ints are detectable by having a pre-defined invalid digit pointer **dp** setting.

Once an mp_int has been cleared the mp_int structure is no longer in a valid state for any other algorithm with the exception of algorithms mp_init, mp_init_copy, mp_init_size and mp_clear.

File: bn_mp_clear.c

```
001   #ifdef BN_MP_CLEAR_C
...
```

The algorithm only operates on the mp_int if it hasn't been previously cleared. The if statement (line 21) checks to see if the **dp** member is not **NULL**. If the mp_int is a valid mp_int then **dp** cannot be **NULL** in which case the if statement will evaluate to true.

The digits of the mp_int are cleared by the for loop (line 23) which assigns a zero to every digit. Similar to mp_init() the digits are assigned zero instead of using block memory operations (such as memset()) since this is more portable.

The digits are deallocated off the heap via the XFREE macro. Similar to XMALLOC the XFREE macro actually evaluates to a standard C library function. In this case the free() function. Since free() only deallocates the memory the pointer still has to be reset to **NULL** manually (line 31).

Now that the digits have been cleared and deallocated the other members are set to their final values (lines 32 and 33).

2.6 Maintenance Algorithms

The previous sections describes how to initialize and clear an mp_int structure. To further support operations that are to be performed on mp_int structures (such as addition and multiplication) the dependent algorithms must be able to augment the precision of an mp_int and initialize mp_ints with differing initial conditions.

These algorithms complete the set of low level algorithms required to work with mp_int structures in the higher level algorithms such as addition, multiplication and modular exponentiation.

2.6.1 Augmenting an mp_int's Precision

When storing a value in an mp_int structure, a sufficient number of digits must be available to accomodate the entire result of an operation without loss of precision. Quite often the size of the array given by the **alloc** member is large enough to simply increase the **used** digit count. However, when the size of the array is too small it must be re-sized appropriately to accomodate the result. The mp_grow algorithm will provide this functionality.

Algorithm **mp_grow**.
Input. An mp_int a and an integer b.
Output. a is expanded to accomodate b digits.

1. if $a.alloc \ge b$ then return(MP_OKAY)
2. $u \leftarrow b \pmod{MP_PREC}$
3. $v \leftarrow b + 2 \cdot MP_PREC - u$
4. Re-allocate the array of digits a to size v
5. If the allocation failed then return(MP_MEM).
6. for n from a.alloc to $v - 1$ do
   6.1 $a_n \leftarrow 0$
7. $a.alloc \leftarrow v$
8. Return(MP_OKAY)

Figure 2.6: Algorithm mp_grow

Algorithm mp_grow. It is ideal to prevent re-allocations from being performed if they are not required (step one). This is useful to prevent mp_ints from growing excessively in code that erroneously calls mp_grow.

The requested digit count is padded up to next multiple of **MP_PREC** plus an additional **MP_PREC** (steps two and three). This helps prevent many trivial reallocations that would grow an mp_int by trivially small values.

It is assumed that the reallocation (step four) leaves the lower $a.alloc$ digits of the mp_int intact. This is much akin to how the *realloc* function from the standard C library works. Since the newly allocated digits are assumed to contain undefined values they are initially set to zero.

File: bn_mp_grow.c

```
001   #ifdef BN_MP_GROW_C
...
```

A quick optimization is to first determine if a memory re-allocation is required at all. The if statement (line 25) checks if the **alloc** member of the mp_int is smaller than the requested digit count. If the count is not larger than **alloc** the function skips the re-allocation part thus saving time.

When a re-allocation is performed it is turned into an optimal request to save time in the future. The requested digit count is padded upwards to 2nd multiple of **MP_PREC** larger than **alloc** (line 25). The XREALLOC function is used to re-allocate the memory. As per the other functions XREALLOC is

actually a macro which evaluates to realloc by default. The realloc function leaves the base of the allocation intact which means the first **alloc** digits of the mp_int are the same as before the re-allocation. All that is left is to clear the newly allocated digits and return.

Note that the re-allocation result is actually stored in a temporary pointer *tmp*. This is to allow this function to return an error with a valid pointer. Earlier releases of the library stored the result of XREALLOC into the mp_int *a*. That would result in a memory leak if XREALLOC ever failed.

2.6.2 Initializing Variable Precision mp_ints

Occasionally the number of digits required will be known in advance of an initialization, based on, for example, the size of input mp_ints to a given algorithm. The purpose of algorithm mp_init_size is similar to mp_init except that it will allocate *at least* a specified number of digits.

Algorithm **mp_init_size**.
Input. An mp_int a and the requested number of digits b.
Output. a is initialized to hold at least b digits.

1. $u \leftarrow b \pmod{MP_PREC}$
2. $v \leftarrow b + 2 \cdot MP_PREC - u$
3. Allocate v digits.
4. for n from 0 to $v - 1$ do
   4.1 $a_n \leftarrow 0$
5. $a.sign \leftarrow MP_ZPOS$
6. $a.used \leftarrow 0$
7. $a.alloc \leftarrow v$
8. Return(MP_OKAY)

Figure 2.7: Algorithm mp_init_size

Algorithm mp_init_size. This algorithm will initialize an mp_int structure a like algorithm mp_init with the exception that the number of digits allocated can be controlled by the second input argument b. The input size is padded upwards so it is a multiple of **MP_PREC** plus an additional **MP_PREC** digits. This padding is used to prevent trivial allocations from becoming a bottleneck in the rest of the algorithms.

Like algorithm mp_init, the mp_int structure is initialized to a default state representing the integer zero. This particular algorithm is useful if it is known

ahead of time the approximate size of the input. If the approximation is correct no further memory re-allocations are required to work with the mp_int.

File: bn_mp_init_size.c

```
001   #ifdef BN_MP_INIT_SIZE_C
...
```

The number of digits b requested is padded (line 25) by first augmenting it to the next multiple of **MP_PREC** and then adding **MP_PREC** to the result. If the memory can be successfully allocated the mp_int is placed in a default state representing the integer zero. Otherwise, the error code **MP_MEM** will be returned (line 30).

The digits are allocated with the malloc() function (line 28) and set to zero afterwards (line 39). The **used** count is set to zero, the **alloc** count set to the padded digit count and the **sign** flag set to **MP_ZPOS** to achieve a default valid mp_int state (lines 34, 27 and 36). If the function returns succesfully then it is correct to assume that the mp_int structure is in a valid state for the remainder of the functions to work with.

2.6.3 Multiple Integer Initializations and Clearings

Occasionally a function will require a series of mp_int data types to be made available simultaneously. The purpose of algorithm mp_init_multi is to initialize a variable length array of mp_int structures in a single statement. It is essentially a shortcut to multiple initializations.

Algorithm **mp_init_multi**.
Input. Variable length array V_k of mp_int variables of length k.
Output. The array is initialized such that each mp_int of V_k is ready to use.

1. for n from 0 to $k-1$ do
   1.1. Initialize the mp_int V_n (*mp_init*)
   1.2. If initialization failed then do
      1.2.1. for j from 0 to n do
         1.2.1.1. Free the mp_int V_j (*mp_clear*)
      1.2.2. Return(*MP_MEM*)
2. Return(*MP_OKAY*)

Figure 2.8: Algorithm mp_init_multi

Algorithm mp_init_multi. The algorithm will initialize the array of mp_int variables one at a time. If a runtime error has been detected (*step 1.2*) all of the previously initialized variables are cleared. The goal is an "all or nothing" initialization which allows for quick recovery from runtime errors.

File: bn_mp_init_multi.c

```
001   #ifdef BN_MP_INIT_MULTI_C
...
```

This function intializes a variable length list of mp_int structure pointers. However, instead of having the mp_int structures in an actual C array they are simply passed as arguments to the function. This function makes use of the "..." argument syntax of the C programming language. The list is terminated with a final **NULL** argument appended on the right.

The function uses the "stdarg.h" *va* functions to step portably through the arguments to the function. A count n of succesfully initialized mp_int structures is maintained (line 43) such that if a failure does occur, the algorithm can backtrack and free the previously initialized structures (lines 26 to 45).

2.6.4 Clamping Excess Digits

When a function anticipates a result will be n digits it is simpler to assume this is true within the body of the function instead of checking during the computation. For example, a multiplication of a i digit number by a j digit produces a result of at most $i+j$ digits. It is entirely possible that the result is $i+j-1$ though,

with no final carry into the last position. However, suppose the destination had to be first expanded (*via mp_grow*) to accomodate $i + j - 1$ digits than further expanded to accomodate the final carry. That would be a considerable waste of time since heap operations are relatively slow.

The ideal solution is to always assume the result is $i + j$ and fix up the **used** count after the function terminates. This way a single heap operation (*at most*) is required. However, if the result was not checked there would be an excess high order zero digit.

For example, suppose the product of two integers was $x_n = (0x_{n-1}x_{n-2}...x_0)_\beta$. The leading zero digit will not contribute to the precision of the result. In fact, through subsequent operations more leading zero digits would accumulate to the point the size of the integer would be prohibitive. As a result even though the precision is very low the representation is excessively large.

The mp_clamp algorithm is designed to solve this very problem. It will trim high-order zeros by decrementing the **used** count until a non-zero most significant digit is found. Also in this system, zero is considered to be a positive number which means that if the **used** count is decremented to zero, the sign must be set to **MP_ZPOS**.

Algorithm **mp_clamp**.
Input. An mp_int a
Output. Any excess leading zero digits of a are removed

1. while $a.used > 0$ and $a_{a.used-1} = 0$ do
   1.1 $a.used \leftarrow a.used - 1$
2. if $a.used = 0$ then do
   2.1 $a.sign \leftarrow MP_ZPOS$

Figure 2.9: Algorithm mp_clamp

Algorithm mp_clamp. As can be expected this algorithm is very simple. The loop on step one is expected to iterate only once or twice at the most. For example, this will happen in cases where there is not a carry to fill the last position. Step two fixes the sign for when all of the digits are zero to ensure that the mp_int is valid at all times.

File: bn_mp_clamp.c

```
001   #ifdef BN_MP_CLAMP_C
```

...

Note on line 27 how to test for the **used** count is made on the left of the && operator. In the C programming language the terms to && are evaluated left to right with a boolean short-circuit if any condition fails. This is important since if the **used** is zero the test on the right would fetch below the array. That is obviously undesirable. The parenthesis on line 28 is used to make sure the **used** count is decremented and not the pointer "a".

Exercises

[1] Discuss the relevance of the **used** member of the mp_int structure.

[1] Discuss the consequences of not using padding when performing allocations.

[2] Estimate an ideal value for **MP_PREC** when performing 1024-bit RSA encryption when $\beta = 2^{28}$.

[1] Discuss the relevance of the algorithm mp_clamp. What does it prevent?

[1] Give an example of when the algorithm mp_init_copy might be useful.

Chapter 3

Basic Operations

3.1 Introduction

In the previous chapter a series of low level algorithms were established that dealt with initializing and maintaining mp_int structures. This chapter will discuss another set of seemingly non-algebraic algorithms which will form the low level basis of the entire library. While these algorithm are relatively trivial it is important to understand how they work before proceeding since these algorithms will be used almost intrinsically in the following chapters.

The algorithms in this chapter deal primarily with more "programmer" related tasks such as creating copies of mp_int structures, assigning small values to mp_int structures and comparisons of the values mp_int structures represent.

3.2 Assigning Values to mp_int Structures

3.2.1 Copying an mp_int

Assigning the value that a given mp_int structure represents to another mp_int structure shall be known as making a copy for the purposes of this text. The copy of the mp_int will be a separate entity that represents the same value as the mp_int it was copied from. The mp_copy algorithm provides this functionality.

Algorithm **mp_copy**.
Input. An mp_int a and b.
Output. Store a copy of a in b.

1. If $b.alloc < a.used$ then grow b to $a.used$ digits. (mp_grow)
2. for n from 0 to $a.used - 1$ do
   2.1 $b_n \leftarrow a_n$
3. for n from $a.used$ to $b.used - 1$ do
   3.1 $b_n \leftarrow 0$
4. $b.used \leftarrow a.used$
5. $b.sign \leftarrow a.sign$
6. return(MP_OKAY)

Figure 3.1: Algorithm mp_copy

Algorithm mp_copy. This algorithm copies the mp_int a such that upon succesful termination of the algorithm the mp_int b will represent the same integer as the mp_int a. The mp_int b shall be a complete and distinct copy of the mp_int a meaing that the mp_int a can be modified and it shall not affect the value of the mp_int b.

If b does not have enough room for the digits of a it must first have its precision augmented via the mp_grow algorithm. The digits of a are copied over the digits of b and any excess digits of b are set to zero (step two and three). The **used** and **sign** members of a are finally copied over the respective members of b.

Remark. This algorithm also introduces a new idiosyncrasy that will be used throughout the rest of the text. The error return codes of other algorithms are not explicitly checked in the pseudo-code presented. For example, in step one of the mp_copy algorithm the return of mp_grow is not explicitly checked to ensure it succeeded. Text space is limited so it is assumed that if a algorithm fails it will clear all temporarily allocated mp_ints and return the error code itself. However, the C code presented will demonstrate all of the error handling logic required to implement the pseudo-code.

File: bn_mp_copy.c
```
001   #ifdef BN_MP_COPY_C
```
...

Occasionally a dependent algorithm may copy an mp_int effectively into itself such as when the input and output mp_int structures passed to a function are

one and the same. For this case it is optimal to return immediately without copying digits (line 21).

The mp_int b must have enough digits to accomodate the used digits of the mp_int a. If $b.alloc$ is less than $a.used$ the algorithm mp_grow is used to augment the precision of b (lines 26 to 30). In order to simplify the inner loop that copies the digits from a to b, two aliases *tmpa* and *tmpb* point directly at the digits of the mp_ints a and b respectively. These aliases (lines 39 and 46) allow the compiler to access the digits without first dereferencing the mp_int pointers and then subsequently the pointer to the digits.

After the aliases are established the digits from a are copied into b (lines 50 to 52) and then the excess digits of b are set to zero (lines 50 to 53). Both "for" loops make use of the pointer aliases and in fact the alias for b is carried through into the second "for" loop to clear the excess digits. This optimization allows the alias to stay in a machine register fairly easy between the two loops.

Remarks. The use of pointer aliases is an implementation methodology first introduced in this function that will be used considerably in other functions. Technically, a pointer alias is simply a short hand alias used to lower the number of pointer dereferencing operations required to access data. For example, a for loop may resemble

```
for (x = 0; x < 100; x++) {
    a->num[4]->dp[x] = 0;
}
```

This could be re-written using aliases as

```
mp_digit *tmpa;
a = a->num[4]->dp;
for (x = 0; x < 100; x++) {
    *a++ = 0;
}
```

In this case an alias is used to access the array of digits within an mp_int structure directly. It may seem that a pointer alias is strictly not required as a compiler may optimize out the redundant pointer operations. However, there are two dominant reasons to use aliases.

The first reason is that most compilers will not effectively optimize pointer arithmetic. For example, some optimizations may work for the Microsoft Visual C++ compiler (MSVC) and not for the GNU C Compiler (GCC). Also some optimizations may work for GCC and not MSVC. As such it is ideal to find a

common ground for as many compilers as possible. Pointer aliases optimize the code considerably before the compiler even reads the source code which means the end compiled code stands a better chance of being faster.

The second reason is that pointer aliases often can make an algorithm simpler to read. Consider the first "for" loop of the function mp_copy() re-written to not use pointer aliases.

```
/* copy all the digits */
for (n = 0; n < a->used; n++) {
  b->dp[n] = a->dp[n];
}
```

Whether this code is harder to read depends strongly on the individual. However, it is quantifiably slightly more complicated as there are four variables within the statement instead of just two.

Nested Statements

Another commonly used technique in the source routines is that certain sections of code are nested. This is used in particular with the pointer aliases to highlight code phases. For example, a Comba multiplier (discussed in chapter six) will typically have three different phases. First the temporaries are initialized, then the columns calculated and finally the carries are propagated. In this example the middle column production phase will typically be nested as it uses temporary variables and aliases the most.

The nesting also simplies the source code as variables that are nested are only valid for their scope. As a result the various temporary variables required do not propagate into other sections of code.

3.2.2 Creating a Clone

Another common operation is to make a local temporary copy of an mp_int argument. To initialize an mp_int and then copy another existing mp_int into the newly intialized mp_int will be known as creating a clone. This is useful within functions that need to modify an argument but do not wish to actually modify the original copy. The mp_init_copy algorithm has been designed to help perform this task.

Algorithm mp_init_copy. This algorithm will initialize an mp_int variable and copy another previously initialized mp_int variable into it. As such this algorithm will perform two operations in one step.

Algorithm **mp_init_copy**.
Input. An mp_int a and b
Output. a is initialized to be a copy of b.

1. Init a. (*mp_init*)
2. Copy b to a. (*mp_copy*)
3. Return the status of the copy operation.

Figure 3.2: Algorithm mp_init_copy

File: bn_mp_init_copy.c

```
001   #ifdef BN_MP_INIT_COPY_C
...
```

This will initialize **a** and make it a verbatim copy of the contents of **b**. Note that **a** will have its own memory allocated which means that **b** may be cleared after the call and **a** will be left intact.

3.3 Zeroing an Integer

Reseting an mp_int to the default state is a common step in many algorithms. The mp_zero algorithm will be the algorithm used to perform this task.

Algorithm **mp_zero**.
Input. An mp_int a
Output. Zero the contents of a

1. $a.used \leftarrow 0$
2. $a.sign \leftarrow$ MP_ZPOS
3. for n from 0 to $a.alloc - 1$ do
   3.1 $a_n \leftarrow 0$

Figure 3.3: Algorithm mp_zero

Algorithm mp_zero. This algorithm simply resets a mp_int to the default state.

File: bn_mp_zero.c

```
001   #ifdef BN_MP_ZERO_C
...
```

After the function is completed, all of the digits are zeroed, the **used** count is zeroed and the **sign** variable is set to **MP_ZPOS**.

3.4 Sign Manipulation

3.4.1 Absolute Value

With the mp_int representation of an integer, calculating the absolute value is trivial. The mp_abs algorithm will compute the absolute value of an mp_int.

Algorithm **mp_abs**.
Input. An mp_int a
Output. Computes $b = |a|$

1. Copy a to b. (mp_copy)
2. If the copy failed return(MP_MEM).
3. $b.sign \leftarrow MP_ZPOS$
4. Return(MP_OKAY)

Figure 3.4: Algorithm mp_abs

Algorithm mp_abs. This algorithm computes the absolute of an mp_int input. First it copies a over b. This is an example of an algorithm where the check in mp_copy that determines if the source and destination are equal proves useful. This allows, for instance, the developer to pass the same mp_int as the source and destination to this function without addition logic to handle it.

File: bn_mp_abs.c
```
001   #ifdef BN_MP_ABS_C
...
```

This fairly trivial algorithm first eliminates non–required duplications (line 24) and then sets the **sign** flag to **MP_ZPOS**.

3.4.2 Integer Negation

With the mp_int representation of an integer, calculating the negation is also trivial. The mp_neg algorithm will compute the negative of an mp_int input.

Algorithm **mp_neg**.
Input. An mp_int a
Output. Computes $b = -a$

1. Copy a to b. (mp_copy)
2. If the copy failed return(MP_MEM).
3. If $a.used = 0$ then return(MP_OKAY).
4. If $a.sign = MP_ZPOS$ then do
   4.1 $b.sign = MP_NEG$.
5. else do
   5.1 $b.sign = MP_ZPOS$.
6. Return(MP_OKAY)

Figure 3.5: Algorithm mp_neg

Algorithm mp_neg. This algorithm computes the negation of an input. First it copies a over b. If a has no used digits then the algorithm returns immediately. Otherwise it flips the sign flag and stores the result in b. Note that if a had no digits then it must be positive by definition. Had step three been omitted then the algorithm would return zero as negative.

File: bn_mp_neg.c

```
001   #ifdef BN_MP_NEG_C
...
```

Like mp_abs() this function avoids non–required duplications (line 19) and then sets the sign. We have to make sure that only non–zero values get a **sign** of **MP_NEG**. If the mp_int is zero than the **sign** is hard–coded to **MP_ZPOS**.

3.5 Small Constants

3.5.1 Setting Small Constants

Often a mp_int must be set to a relatively small value such as 1 or 2. For these cases the mp_set algorithm is useful.

Algorithm **mp_set**.
Input. An mp_int a and a digit b
Output. Make a equivalent to b

1. Zero a (*mp_zero*).
2. $a_0 \leftarrow b \pmod{\beta}$
3. $a.used \leftarrow \begin{cases} 1 & \text{if } a_0 > 0 \\ 0 & \text{if } a_0 = 0 \end{cases}$

Figure 3.6: Algorithm mp_set

Algorithm mp_set. This algorithm sets a mp_int to a small single digit value. Step number 1 ensures that the integer is reset to the default state. The single digit is set (*modulo* β) and the **used** count is adjusted accordingly.

File: bn_mp_set.c

```
001   #ifdef BN_MP_SET_C
...
```

First we zero (line 18) the mp_int to make sure that the other members are initialized for a small positive constant. mp_zero() ensures that the **sign** is positive and the **used** count is zero. Next we set the digit and reduce it modulo β (line 19). After this step we have to check if the resulting digit is zero or not. If it is not then we set the **used** count to one, otherwise to zero.

We can quickly reduce modulo β since it is of the form 2^k and a quick binary AND operation with $2^k - 1$ will perform the same operation.

One important limitation of this function is that it will only set one digit. The size of a digit is not fixed, meaning source that uses this function should take that into account. Only trivially small constants can be set using this function.

3.5.2 Setting Large Constants

To overcome the limitations of the mp_set algorithm the mp_set_int algorithm is ideal. It accepts a "long" data type as input and will always treat it as a 32-bit integer.

Algorithm mp_set_int. The algorithm performs eight iterations of a simple loop where in each iteration four bits from the source are added to the mp_int. Step 2.1 will multiply the current result by sixteen making room for

Algorithm **mp_set_int**.
Input. An mp_int a and a "long" integer b
Output. Make a equivalent to b

1. Zero a (*mp_zero*)
2. for n from 0 to 7 do
   2.1 $a \leftarrow a \cdot 16$ (*mp_mul2d*)
   2.2 $u \leftarrow \lfloor b/2^{4(7-n)} \rfloor \pmod{16}$
   2.3 $a_0 \leftarrow a_0 + u$
   2.4 $a.used \leftarrow a.used + 1$
3. Clamp excess used digits (*mp_clamp*)

Figure 3.7: Algorithm mp_set_int

four more bits in the less significant positions. In step 2.2 the next four bits from the source are extracted and are added to the mp_int. The **used** digit count is incremented to reflect the addition. The **used** digit counter is incremented since if any of the leading digits were zero the mp_int would have zero digits used and the newly added four bits would be ignored.

Excess zero digits are trimmed in steps 2.1 and 3 by using higher level algorithms mp_mul2d and mp_clamp.

File: bn_mp_set_int.c
```
001   #ifdef BN_MP_SET_INT_C
```
...

This function sets four bits of the number at a time to handle all practical **DIGIT_BIT** sizes. The weird addition on line 36 ensures that the newly added in bits are added to the number of digits. While it may not seem obvious as to why the digit counter does not grow exceedingly large it is because of the shift on line 25 as well as the call to mp_clamp() on line 38. Both functions will clamp excess leading digits which keeps the number of used digits low.

3.6 Comparisons

3.6.1 Unsigned Comparisions

Comparing a multiple precision integer is performed with the exact same algorithm used to compare two decimal numbers. For example, to compare

$1,234$ to $1,264$ the digits are extracted by their positions. That is we compare $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$ to $1 \cdot 10^3 + 2 \cdot 10^2 + 6 \cdot 10^1 + 4 \cdot 10^0$ by comparing single digits at a time starting with the highest magnitude positions. If any leading digit of one integer is greater than a digit in the same position of another integer then obviously it must be greater.

The first comparision routine that will be developed is the unsigned magnitude compare which will perform a comparison based on the digits of two mp_int variables alone. It will ignore the sign of the two inputs. Such a function is useful when an absolute comparison is required or if the signs are known to agree in advance.

To facilitate working with the results of the comparison functions three constants are required.

Constant	Meaning
MP_GT	Greater Than
MP_EQ	Equal To
MP_LT	Less Than

Figure 3.8: Comparison Return Codes

Algorithm **mp_cmp_mag**.
Input. Two mp_ints a and b.
Output. Unsigned comparison results (a to the left of b).

1. If $a.used > b.used$ then return(MP_GT)
2. If $a.used < b.used$ then return(MP_LT)
3. for n from $a.used - 1$ to 0 do
   3.1 if $a_n > b_n$ then return(MP_GT)
   3.2 if $a_n < b_n$ then return(MP_LT)
4. Return(MP_EQ)

Figure 3.9: Algorithm mp_cmp_mag

Algorithm mp_cmp_mag. By saying "a to the left of b" it is meant that the comparison is with respect to a, that is if a is greater than b it will return **MP_GT** and similar with respect to when $a = b$ and $a < b$. The first two steps compare the number of digits used in both a and b. Obviously if the

digit counts differ there would be an imaginary zero digit in the smaller number where the leading digit of the larger number is. If both have the same number of digits than the actual digits themselves must be compared starting at the leading digit.

By step three both inputs must have the same number of digits so its safe to start from either $a.used - 1$ or $b.used - 1$ and count down to the zero'th digit. If after all of the digits have been compared, no difference is found, the algorithm returns **MP_EQ**.

File: bn_mp_cmp_mag.c

```
001   #ifdef BN_MP_CMP_MAG_C
...
```

The two if statements (lines 26 and 26) compare the number of digits in the two inputs. These two are performed before all of the digits are compared since it is a very cheap test to perform and can potentially save considerable time. The implementation given is also not valid without those two statements. $b.alloc$ may be smaller than $a.used$, meaning that undefined values will be read from b past the end of the array of digits.

3.6.2 Signed Comparisons

Comparing with sign considerations is also fairly critical in several routines (*division for example*). Based on an unsigned magnitude comparison a trivial signed comparison algorithm can be written.

Algorithm **mp_cmp**.
Input. Two mp_ints a and b
Output. Signed Comparison Results (a to the left of b)

1. if $a.sign = MP_NEG$ and $b.sign = MP_ZPOS$ then return(MP_LT)
2. if $a.sign = MP_ZPOS$ and $b.sign = MP_NEG$ then return(MP_GT)
3. if $a.sign = MP_NEG$ then
   3.1 Return the unsigned comparison of b and a (*mp_cmp_mag*)
4 Otherwise
   4.1 Return the unsigned comparison of a and b

Figure 3.10: Algorithm mp_cmp

Algorithm mp_cmp. The first two steps compare the signs of the two inputs. If the signs do not agree then it can return right away with the appropriate comparison code. When the signs are equal the digits of the inputs must be compared to determine the correct result. In step three the unsigned comparision flips the order of the arguments since they are both negative. For instance, if $-a > -b$ then $|a| < |b|$. Step number four will compare the two when they are both positive.

File: bn_mp_cmp.c

```
001   #ifdef BN_MP_CMP_C
...
```

The two if statements (lines 20 and 28) perform the initial sign comparison. If the signs are not the equal then which ever has the positive sign is larger. The inputs are compared (line 29) based on magnitudes. If the signs were both negative then the unsigned comparison is performed in the opposite direction (line 32). Otherwise, the signs are assumed to be both positive and a forward direction unsigned comparison is performed.

Exercises

[2] Modify algorithm mp_set_int to accept as input a variable length array of bits.

[3] Give the probability that algorithm mp_cmp_mag will have to compare k digits of two random digits (of equal magnitude) before a difference is found.

[1] Suggest a simple method to speed up the implementation of mp_cmp_mag based on the observations made in the previous problem.

Chapter 4

Basic Arithmetic

4.1 Introduction

At this point algorithms for initialization, clearing, zeroing, copying, comparing and setting small constants have been established. The next logical set of algorithms to develop are addition, subtraction and digit shifting algorithms. These algorithms make use of the lower level algorithms and are the cruicial building block for the multiplication algorithms. It is very important that these algorithms are highly optimized. On their own they are simple $O(n)$ algorithms but they can be called from higher level algorithms which easily places them at $O(n^2)$ or even $O(n^3)$ work levels.

All of the algorithms within this chapter make use of the logical bit shift operations denoted by $<<$ and $>>$ for left and right logical shifts respectively. A logical shift is analogous to sliding the decimal point of radix-10 representations. For example, the real number 0.9345 is equivalent to 93.45% which is found by sliding the the decimal two places to the right (*multiplying by* $\beta^2 = 10^2$). Algebraically a binary logical shift is equivalent to a division or multiplication by a power of two. For example, $a << k = a \cdot 2^k$ while $a >> k = \lfloor a/2^k \rfloor$.

One significant difference between a logical shift and the way decimals are shifted is that digits below the zero'th position are removed from the number. For example, consider $1101_2 >> 1$ using decimal notation this would produce 110.1_2. However, with a logical shift the result is 110_2.

4.2 Addition and Subtraction

In common twos complement fixed precision arithmetic negative numbers are easily represented by subtraction from the modulus. For example, with 32-bit integers $a - b \pmod{2^{32}}$ is the same as $a + (2^{32} - b) \pmod{2^{32}}$ since $2^{32} \equiv 0 \pmod{2^{32}}$. As a result subtraction can be performed with a trivial series of logical operations and an addition.

However, in multiple precision arithmetic negative numbers are not represented in the same way. Instead a sign flag is used to keep track of the sign of the integer. As a result signed addition and subtraction are actually implemented as conditional usage of lower level addition or subtraction algorithms with the sign fixed up appropriately.

The lower level algorithms will add or subtract integers without regard to the sign flag. That is they will add or subtract the magnitude of the integers respectively.

4.2.1 Low Level Addition

An unsigned addition of multiple precision integers is performed with the same long-hand algorithm used to add decimal numbers. That is to add the trailing digits first and propagate the resulting carry upwards. Since this is a lower level algorithm the name will have a "s_" prefix. Historically that convention stems from the MPI library where "s_" stood for static functions that were hidden from the developer entirely.

Algorithm **s_mp_add**.
Input. Two mp_ints a and b
Output. The unsigned addition $c = |a| + |b|$.

1. if $a.used > b.used$ then
   1.1 $min \leftarrow b.used$
   1.2 $max \leftarrow a.used$
   1.3 $x \leftarrow a$
2. else
   2.1 $min \leftarrow a.used$
   2.2 $max \leftarrow b.used$
   2.3 $x \leftarrow b$
3. If $c.alloc < max + 1$ then grow c to hold at least $max + 1$ digits (mp_grow)
4. $oldused \leftarrow c.used$
5. $c.used \leftarrow max + 1$
6. $u \leftarrow 0$
7. for n from 0 to $min - 1$ do
   7.1 $c_n \leftarrow a_n + b_n + u$
   7.2 $u \leftarrow c_n >> lg(\beta)$
   7.3 $c_n \leftarrow c_n \pmod{\beta}$
8. if $min \neq max$ then do
   8.1 for n from min to $max - 1$ do
      8.1.1 $c_n \leftarrow x_n + u$
      8.1.2 $u \leftarrow c_n >> lg(\beta)$
      8.1.3 $c_n \leftarrow c_n \pmod{\beta}$
9. $c_{max} \leftarrow u$
10. if $olduse > max$ then
   10.1 for n from $max + 1$ to $oldused - 1$ do
      10.1.1 $c_n \leftarrow 0$
11. Clamp excess digits in c. (mp_clamp)
12. Return(MP_OKAY)

Figure 4.1: Algorithm s_mp_add

Algorithm s_mp_add. This algorithm is loosely based on algorithm 14.7 of HAC [2, pp. 594] but has been extended to allow the inputs to have different magnitudes. Coincidentally the description of algorithm A in Knuth [1, pp. 266] shares the same deficiency as the algorithm from [2]. Even the MIX pseudo machine code presented by Knuth [1, pp. 266-267] is incapable of handling inputs which are of different magnitudes.

The first thing that has to be accomplished is to sort out which of the two

inputs is the largest. The addition logic will simply add all of the smallest input to the largest input and store that first part of the result in the destination. Then it will apply a simpler addition loop to excess digits of the larger input.

The first two steps will handle sorting the inputs such that min and max hold the digit counts of the two inputs. The variable x will be an mp_int alias for the largest input or the second input b if they have the same number of digits. After the inputs are sorted the destination c is grown as required to accomodate the sum of the two inputs. The original **used** count of c is copied and set to the new used count.

At this point the first addition loop will go through as many digit positions that both inputs have. The carry variable μ is set to zero outside the loop. Inside the loop an "addition" step requires three statements to produce one digit of the summand. First two digits from a and b are added together along with the carry μ. The carry of this step is extracted and stored in μ and finally the digit of the result c_n is truncated within the range $0 \le c_n < \beta$.

Now all of the digit positions that both inputs have in common have been exhausted. If $min \neq max$ then x is an alias for one of the inputs that has more digits. A simplified addition loop is then used to essentially copy the remaining digits and the carry to the destination.

The final carry is stored in c_{max} and digits above max upto $oldused$ are zeroed which completes the addition.

File: bn_s_mp_add.c

```
001   #ifdef BN_S_MP_ADD_C
...
```

We first sort (lines 24 to 38) the inputs based on magnitude and determine the min and max variables. Note that x is a pointer to an mp_int assigned to the largest input, in effect it is a local alias. Next we grow the destination (34 to 39) ensure that it can accomodate the result of the addition.

Similar to the implementation of mp_copy this function uses the braced code and local aliases coding style. The three aliases that are on lines 52, 55 and 64 represent the two inputs and destination variables respectively. These aliases are used to ensure the compiler does not have to dereference a, b or c (respectively) to access the digits of the respective mp_int.

The initial carry u will be cleared (line 61), note that u is of type mp_digit which ensures type compatibility within the implementation. The initial addition (line 62 to 71) adds digits from both inputs until the smallest input runs out of digits. Similarly the conditional addition loop (line 77 to 87) adds the remaining digits from the larger of the two inputs. The addition is finished with

the final carry being stored in *tmpc* (line 94). Note the "++" operator within the same expression. After line 94, *tmpc* will point to the *c.used*'th digit of the mp_int *c*. This is useful for the next loop (line 93 to 100) which set any old upper digits to zero.

4.2.2 Low Level Subtraction

The low level unsigned subtraction algorithm is very similar to the low level unsigned addition algorithm. The principle difference is that the unsigned subtraction algorithm requires the result to be positive. That is when computing $a - b$ the condition $|a| \ge |b|$ must be met for this algorithm to function properly. Keep in mind this low level algorithm is not meant to be used in higher level algorithms directly. This algorithm as will be shown can be used to create functional signed addition and subtraction algorithms.

For this algorithm a new variable is required to make the description simpler. Recall from section 1.3.1 that a mp_digit must be able to represent the range $0 \le x < 2\beta$ for the algorithms to work correctly. However, it is allowable that a mp_digit represent a larger range of values. For this algorithm we will assume that the variable γ represents the number of bits available in a mp_digit (*this implies* $2^\gamma > \beta$).

For example, the default for LibTomMath is to use a "unsigned long" for the mp_digit "type" while $\beta = 2^{28}$. In ISO C an "unsigned long" data type must be able to represent $0 \le x < 2^{32}$ meaning that in this case $\gamma \ge 32$.

Algorithm **s_mp_sub**.
Input. Two mp_ints a and b ($|a| \geq |b|$)
Output. The unsigned subtraction $c = |a| - |b|$.

1. $min \leftarrow b.used$
2. $max \leftarrow a.used$
3. If $c.alloc < max$ then grow c to hold at least max digits. (mp_grow)
4. $oldused \leftarrow c.used$
5. $c.used \leftarrow max$
6. $u \leftarrow 0$
7. for n from 0 to $min - 1$ do
   7.1 $c_n \leftarrow a_n - b_n - u$
   7.2 $u \leftarrow c_n >> (\gamma - 1)$
   7.3 $c_n \leftarrow c_n \pmod{\beta}$
8. if $min < max$ then do
   8.1 for n from min to $max - 1$ do
      8.1.1 $c_n \leftarrow a_n - u$
      8.1.2 $u \leftarrow c_n >> (\gamma - 1)$
      8.1.3 $c_n \leftarrow c_n \pmod{\beta}$
9. if $oldused > max$ then do
   9.1 for n from max to $oldused - 1$ do
      9.1.1 $c_n \leftarrow 0$
10. Clamp excess digits of c. (mp_clamp).
11. Return(MP_OKAY).

Figure 4.2: Algorithm s_mp_sub

Algorithm s_mp_sub. This algorithm performs the unsigned subtraction of two mp_int variables under the restriction that the result must be positive. That is when passing variables a and b the condition that $|a| \geq |b|$ must be met for the algorithm to function correctly. This algorithm is loosely based on algorithm 14.9 [2, pp. 595] and is similar to algorithm S in [1, pp. 267] as well. As was the case of the algorithm s_mp_add both other references lack discussion concerning various practical details such as when the inputs differ in magnitude.

The initial sorting of the inputs is trivial in this algorithm since a is guaranteed to have at least the same magnitude of b. Steps 1 and 2 set the min and max variables. Unlike the addition routine there is guaranteed to be no carry which means that the final result can be at most max digits in length as opposed to $max + 1$. Similar to the addition algorithm the **used** count of c is copied locally and set to the maximal count for the operation.

The subtraction loop that begins on step seven is essentially the same as the addition loop of algorithm s_mp_add except single precision subtraction is used instead. Note the use of the γ variable to extract the carry (*also known as the borrow*) within the subtraction loops. Under the assumption that two's complement single precision arithmetic is used this will successfully extract the desired carry.

For example, consider subtracting 0101_2 from 0100_2 where $\gamma = 4$ and $\beta = 2$. The least significant bit will force a carry upwards to the third bit which will be set to zero after the borrow. After the very first bit has been subtracted $4 - 1 \equiv 0011_2$ will remain, When the third bit of 0101_2 is subtracted from the result it will cause another carry. In this case though the carry will be forced to propagate all the way to the most significant bit.

Recall that $\beta < 2^\gamma$. This means that if a carry does occur just before the $lg(\beta)$'th bit it will propagate all the way to the most significant bit. Thus, the high order bits of the mp_digit that are not part of the actual digit will either be all zero, or all one. All that is needed is a single zero or one bit for the carry. Therefore a single logical shift right by $\gamma - 1$ positions is sufficient to extract the carry. This method of carry extraction may seem awkward but the reason for it becomes apparent when the implementation is discussed.

If b has a smaller magnitude than a then step 9 will force the carry and copy operation to propagate through the larger input a into c. Step 10 will ensure that any leading digits of c above the max'th position are zeroed.

File: bn_s_mp_sub.c

```
001   #ifdef BN_S_MP_SUB_C
...
```

Like low level addition we "sort" the inputs. Except in this case the sorting is hardcoded (lines 21 and 25). In reality the *min* and *max* variables are only aliases and are only used to make the source code easier to read. Again the pointer alias optimization is used within this algorithm. The aliases *tmpa*, *tmpb* and *tmpc* are initialized (lines 46, 46 and 46) for a, b and c respectively.

The first subtraction loop (lines 43 through 57) subtract digits from both inputs until the smaller of the two inputs has been exhausted. As remarked earlier there is an implementation reason for using the "awkward" method of extracting the carry (line 53). The traditional method for extracting the carry would be to shift by $lg(\beta)$ positions and logically AND the least significant bit. The AND operation is required because all of the bits above the lg(β)'th bit will be set to one after a carry occurs from subtraction. This carry extraction requires two relatively cheap operations to extract the carry. The other method

is to simply shift the most significant bit to the least significant bit thus extracting the carry with a single cheap operation. This optimization only works on twos compliment machines which is a safe assumption to make.

If a has a larger magnitude than b an additional loop (lines 60 through 74) is required to propagate the carry through a and copy the result to c.

4.2.3 High Level Addition

Now that both lower level addition and subtraction algorithms have been established an effective high level signed addition algorithm can be established. This high level addition algorithm will be what other algorithms and developers will use to perform addition of mp_int data types.

Recall from section 5.2 that an mp_int represents an integer with an unsigned mantissa (*the array of digits*) and a **sign** flag. A high level addition is actually performed as a series of eight separate cases which can be optimized down to three unique cases.

Algorithm **mp_add**.
Input. Two mp_ints a and b
Output. The signed addition $c = a + b$.

1. if $a.sign = b.sign$ then do
   1.1 $c.sign \leftarrow a.sign$
   1.2 $c \leftarrow |a| + |b|$ (s_mp_add)
2. else do
   2.1 if $|a| < |b|$ then do (mp_cmp_mag)
      2.1.1 $c.sign \leftarrow b.sign$
      2.1.2 $c \leftarrow |b| - |a|$ (s_mp_sub)
   2.2 else do
      2.2.1 $c.sign \leftarrow a.sign$
      2.2.2 $c \leftarrow |a| - |b|$
3. Return(MP_OKAY).

Figure 4.3: Algorithm mp_add

Algorithm mp_add. This algorithm performs the signed addition of two mp_int variables. There is no reference algorithm to draw upon from either [1] or [2] since they both only provide unsigned operations. The algorithm is

fairly straightforward but restricted since subtraction can only produce positive results.

Sign of a	**Sign of** b	$\|a\| > \|b\|$	**Unsigned Operation**	**Result Sign Flag**
+	+	Yes	$c = a + b$	$a.sign$
+	+	No	$c = a + b$	$a.sign$
−	−	Yes	$c = a + b$	$a.sign$
−	−	No	$c = a + b$	$a.sign$
+	−	No	$c = b - a$	$b.sign$
−	+	No	$c = b - a$	$b.sign$
+	−	Yes	$c = a - b$	$a.sign$
−	+	Yes	$c = a - b$	$a.sign$

Figure 4.4: Addition Guide Chart

Figure 4.4 lists all of the eight possible input combinations and is sorted to show that only three specific cases need to be handled. The return code of the unsigned operations at step 1.2, 2.1.2 and 2.2.2 are forwarded to step three to check for errors. This simplifies the description of the algorithm considerably and best follows how the implementation actually was achieved.

Also note how the **sign** is set before the unsigned addition or subtraction is performed. Recall from the descriptions of algorithms s_mp_add and s_mp_sub that the mp_clamp function is used at the end to trim excess digits. The mp_clamp algorithm will set the **sign** to **MP_ZPOS** when the **used** digit count reaches zero.

For example, consider performing $-a + a$ with algorithm mp_add. By the description of the algorithm the sign is set to **MP_NEG** which would produce a result of -0. However, since the sign is set first then the unsigned addition is performed the subsequent usage of algorithm mp_clamp within algorithm s_mp_add will force -0 to become 0.

File: bn_mp_add.c

```
001   #ifdef BN_MP_ADD_C
...
```

The source code follows the algorithm fairly closely. The most notable new source code addition is the usage of the *res* integer variable which is used to pass result of the unsigned operations forward. Unlike in the algorithm, the

variable *res* is merely returned as is without explicitly checking it and returning the constant **MP_OKAY**. The observation is this algorithm will succeed or fail only if the lower level functions do so. Returning their return code is sufficient.

4.2.4 High Level Subtraction

The high level signed subtraction algorithm is essentially the same as the high level signed addition algorithm.

Algorithm **mp_sub**.
Input. Two mp_ints a and b
Output. The signed subtraction $c = a - b$.

1. if $a.sign \neq b.sign$ then do
   1.1 $c.sign \leftarrow a.sign$
   1.2 $c \leftarrow |a| + |b|$ (s_mp_add)
2. else do
   2.1 if $|a| \geq |b|$ then do (mp_cmp_mag)
      2.1.1 $c.sign \leftarrow a.sign$
      2.1.2 $c \leftarrow |a| - |b|$ (s_mp_sub)
   2.2 else do
      2.2.1 $c.sign \leftarrow \begin{cases} MP_ZPOS & \text{if } a.sign = MP_NEG \\ MP_NEG & \text{otherwise} \end{cases}$
      2.2.2 $c \leftarrow |b| - |a|$
3. Return(MP_OKAY).

Figure 4.5: Algorithm mp_sub

Algorithm mp_sub. This algorithm performs the signed subtraction of two inputs. Similar to algorithm mp_add there is no reference in either [1] or [2]. Also this algorithm is restricted by algorithm s_mp_sub. Chart 4.6 lists the eight possible inputs and the operations required.

Sign of a	**Sign of b**	$\|a\| \geq \|b\|$	**Unsigned Operation**	**Result Sign Flag**
+	−	Yes	$c = a + b$	$a.sign$
+	−	No	$c = a + b$	$a.sign$
−	+	Yes	$c = a + b$	$a.sign$
−	+	No	$c = a + b$	$a.sign$
+	+	Yes	$c = a - b$	$a.sign$
−	−	Yes	$c = a - b$	$a.sign$
+	+	No	$c = b - a$	opposite of $a.sign$
−	−	No	$c = b - a$	opposite of $a.sign$

Figure 4.6: Subtraction Guide Chart

Similar to the case of algorithm mp_add the **sign** is set first before the unsigned addition or subtraction. That is to prevent the algorithm from producing $-a - -a = -0$ as a result.

File: bn_mp_sub.c

```
001   #ifdef BN_MP_SUB_C
...
```

Much like the implementation of algorithm mp_add the variable res is used to catch the return code of the unsigned addition or subtraction operations and forward it to the end of the function. On line 35 the "not equal to" **MP_LT** expression is used to emulate a "greater than or equal to" comparison.

4.3 Bit and Digit Shifting

It is quite common to think of a multiple precision integer as a polynomial in x, that is $y = f(\beta)$ where $f(x) = \sum_{i=0}^{n-1} a_i x^i$. This notation arises within discussion of Montgomery and Diminished Radix Reduction as well as Karatsuba multiplication and squaring.

In order to facilitate operations on polynomials in x as above a series of simple "digit" algorithms have to be established. That is to shift the digits left or right as well to shift individual bits of the digits left and right. It is important to note that not all "shift" operations are on radix-β digits.

4.3.1 Multiplication by Two

In a binary system where the radix is a power of two multiplication by two not only arises often in other algorithms it is a fairly efficient operation to perform. A single precision logical shift left is sufficient to multiply a single digit by two.

Algorithm **mp_mul_2**.
Input. One mp_int a
Output. $b = 2a$.

1. If $b.alloc < a.used + 1$ then grow b to hold $a.used + 1$ digits. (mp_grow)
2. $oldused \leftarrow b.used$
3. $b.used \leftarrow a.used$
4. $r \leftarrow 0$
5. for n from 0 to $a.used - 1$ do
   5.1 $rr \leftarrow a_n >> (lg(\beta) - 1)$
   5.2 $b_n \leftarrow (a_n << 1) + r \pmod{\beta}$
   5.3 $r \leftarrow rr$
6. If $r \neq 0$ then do
   6.1 $b_{n+1} \leftarrow r$
   6.2 $b.used \leftarrow b.used + 1$
7. If $b.used < oldused - 1$ then do
   7.1 for n from $b.used$ to $oldused - 1$ do
      7.1.1 $b_n \leftarrow 0$
8. $b.sign \leftarrow a.sign$
9. Return(MP_OKAY).

Figure 4.7: Algorithm mp_mul_2

Algorithm mp_mul_2. This algorithm will quickly multiply a mp_int by two provided β is a power of two. Neither [1] nor [2] describe such an algorithm despite the fact it arises often in other algorithms. The algorithm is setup much like the lower level algorithm s_mp_add since it is for all intents and purposes equivalent to the operation $b = |a| + |a|$.

Step 1 and 2 grow the input as required to accomodate the maximum number of **used** digits in the result. The initial **used** count is set to $a.used$ at step 4. Only if there is a final carry will the **used** count require adjustment.

Step 6 is an optimization implementation of the addition loop for this specific case. That is since the two values being added together are the same there is no need to perform two reads from the digits of a. Step 6.1 performs a single precision shift on the current digit a_n to obtain what will be the carry for the next iteration. Step 6.2 calculates the n'th digit of the result as single precision shift of a_n plus the previous carry. Recall from section 4.1 that $a_n << 1$ is equivalent to $a_n \cdot 2$. An iteration of the addition loop is finished with forwarding the carry to the next iteration.

Step 7 takes care of any final carry by setting the $a.used$'th digit of the

result to the carry and augmenting the **used** count of b. Step 8 clears any leading digits of b in case it originally had a larger magnitude than a.

File: bn_mp_mul_2.c

```
001   #ifdef BN_MP_MUL_2_C
...
```

This implementation is essentially an optimized implementation of s_mp_add for the case of doubling an input. The only noteworthy difference is the use of the logical shift operator on line 49 to perform a single precision doubling.

4.3.2 Division by Two

A division by two can just as easily be accomplished with a logical shift right as multiplication by two can be with a logical shift left.

Algorithm **mp_div_2**.
Input. One mp_int a
Output. $b = a/2$.

1. If $b.alloc < a.used$ then grow b to hold $a.used$ digits. (mp_grow)
2. If the reallocation failed return(MP_MEM).
3. $oldused \leftarrow b.used$
4. $b.used \leftarrow a.used$
5. $r \leftarrow 0$
6. for n from $b.used - 1$ to 0 do
   6.1 $rr \leftarrow a_n \pmod 2$
   6.2 $b_n \leftarrow (a_n >> 1) + (r << (lg(\beta) - 1)) \pmod \beta$
   6.3 $r \leftarrow rr$
7. If $b.used < oldused - 1$ then do
   7.1 for n from $b.used$ to $oldused - 1$ do
      7.1.1 $b_n \leftarrow 0$
8. $b.sign \leftarrow a.sign$
9. Clamp excess digits of b. (mp_clamp)
10. Return(MP_OKAY).

Figure 4.8: Algorithm mp_div_2

Algorithm mp_div_2. This algorithm will divide an mp_int by two using logical shifts to the right. Like mp_mul_2 it uses a modified low level addition core as the basis of the algorithm. Unlike mp_mul_2 the shift operations work from the leading digit to the trailing digit. The algorithm could be written to work from the trailing digit to the leading digit however, it would have to stop one short of $a.used - 1$ digits to prevent reading past the end of the array of digits.

Essentially the loop at step 6 is similar to that of mp_mul_2 except the logical shifts go in the opposite direction and the carry is at the least significant bit not the most significant bit.

File: bn_mp_div_2.c

```
001   #ifdef BN_MP_DIV_2_C
...
```

4.4 Polynomial Basis Operations

Recall from section 4.3 that any integer can be represented as a polynomial in x as $y = f(\beta)$. Such a representation is also known as the polynomial basis [3, pp. 48]. Given such a notation a multiplication or division by x amounts to shifting whole digits a single place. The need for such operations arises in several other higher level algorithms such as Barrett and Montgomery reduction, integer division and Karatsuba multiplication.

Converting from an array of digits to polynomial basis is very simple. Consider the integer $y \equiv (a_2, a_1, a_0)_\beta$ and recall that $y = \sum_{i=0}^{2} a_i \beta^i$. Simply replace β with x and the expression is in polynomial basis. For example, $f(x) = 8x + 9$ is the polynomial basis representation for 89 using radix ten. That is, $f(10) = 8(10) + 9 = 89$.

4.4.1 Multiplication by x

Given a polynomial in x such as $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ multiplying by x amounts to shifting the coefficients up one degree. In this case $f(x) \cdot x = a_n x^{n+1} + a_{n-1} x^n + ... + a_0 x$. From a scalar basis point of view multiplying by x is equivalent to multiplying by the integer β.

Algorithm **mp_lshd**.
Input. One mp_int a and an integer b
Output. $a \leftarrow a \cdot \beta^b$ (equivalent to multiplication by x^b).

1. If $b \leq 0$ then return(MP_OKAY).
2. If $a.alloc < a.used + b$ then grow a to at least $a.used + b$ digits. (mp_grow).
3. If the reallocation failed return(MP_MEM).
4. $a.used \leftarrow a.used + b$
5. $i \leftarrow a.used - 1$
6. $j \leftarrow a.used - 1 - b$
7. for n from $a.used - 1$ to b do
   7.1 $a_i \leftarrow a_j$
   7.2 $i \leftarrow i - 1$
   7.3 $j \leftarrow j - 1$
8. for n from 0 to $b - 1$ do
   8.1 $a_n \leftarrow 0$
9. Return(MP_OKAY).

Figure 4.9: Algorithm mp_lshd

Algorithm mp_lshd. This algorithm multiplies an mp_int by the b'th power of x. This is equivalent to multiplying by β^b. The algorithm differs from the other algorithms presented so far as it performs the operation in place instead storing the result in a separate location. The motivation behind this change is due to the way this function is typically used. Algorithms such as mp_add store the result in an optionally different third mp_int because the original inputs are often still required. Algorithm mp_lshd (*and similarly algorithm mp_rshd*) is typically used on values where the original value is no longer required. The algorithm will return success immediately if $b \leq 0$ since the rest of algorithm is only valid when $b > 0$.

First the destination a is grown as required to accomodate the result. The counters i and j are used to form a *sliding window* over the digits of a of length b. The head of the sliding window is at i (*the leading digit*) and the tail at j (*the trailing digit*). The loop on step 7 copies the digit from the tail to the head. In each iteration the window is moved down one digit. The last loop on step 8 sets the lower b digits to zero.

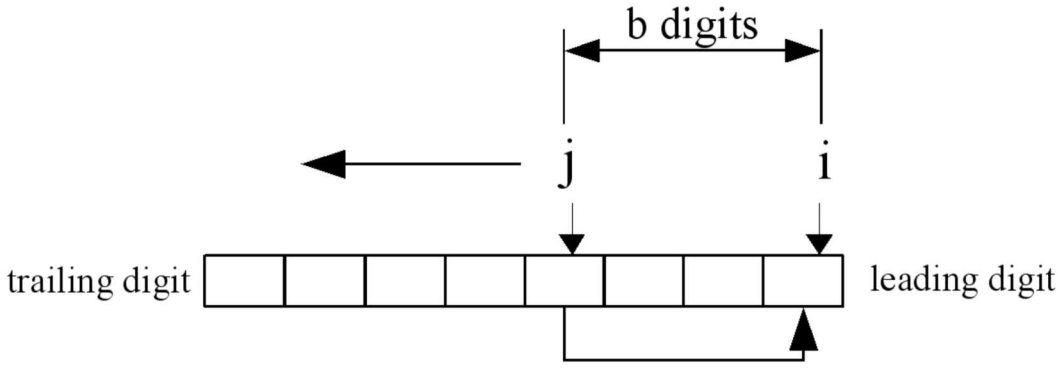


Figure 4.10: Sliding Window Movement

File: bn_mp_lshd.c

```
001   #ifdef BN_MP_LSHD_C
...
```

The if statement (line 24) ensures that the b variable is greater than zero since we do not interpret negative shift counts properly. The **used** count is incremented by b before the copy loop begins. This elminates the need for an additional variable in the for loop. The variable *top* (line 42) is an alias for the leading digit while *bottom* (line 46) is an alias for the trailing edge. The aliases form a window of exactly b digits over the input.

4.4.2 Division by x

Division by powers of x is easily achieved by shifting the digits right and removing any that will end up to the right of the zero'th digit.

Algorithm **mp_rshd**.
Input. One mp_int a and an integer b
Output. $a \leftarrow a/\beta^b$ (Divide by x^b).

1. If $b \leq 0$ then return.
2. If $a.used \leq b$ then do
   2.1 Zero a. (*mp_zero*).
   2.2 Return.
3. $i \leftarrow 0$
4. $j \leftarrow b$
5. for n from 0 to $a.used - b - 1$ do
   5.1 $a_i \leftarrow a_j$
   5.2 $i \leftarrow i + 1$
   5.3 $j \leftarrow j + 1$
6. for n from $a.used - b$ to $a.used - 1$ do
   6.1 $a_n \leftarrow 0$
7. $a.used \leftarrow a.used - b$
8. Return.

Figure 4.11: Algorithm mp_rshd

Algorithm mp_rshd. This algorithm divides the input in place by the b'th power of x. It is analogous to dividing by a β^b but much quicker since it does not require single precision division. This algorithm does not actually return an error code as it cannot fail.

If the input b is less than one the algorithm quickly returns without performing any work. If the **used** count is less than or equal to the shift count b then it will simply zero the input and return.

After the trivial cases of inputs have been handled the sliding window is setup. Much like the case of algorithm mp_lshd a sliding window that is b digits wide is used to copy the digits. Unlike mp_lshd the window slides in the opposite direction from the trailing to the leading digit. Also the digits are copied from the leading to the trailing edge.

Once the window copy is complete the upper digits must be zeroed and the **used** count decremented.

File: bn_mp_rshd.c

```
001   #ifdef BN_MP_RSHD_C
...
```

The only noteworthy element of this routine is the lack of a return type since

it cannot fail. Like mp_lshd() we form a sliding window except we copy in the other direction. After the window (line 57) we then zero the upper digits of the input to make sure the result is correct.

4.5 Powers of Two

Now that algorithms for moving single bits as well as whole digits exist algorithms for moving the "in between" distances are required. For example, to quickly multiply by 2^k for any k without using a full multiplier algorithm would prove useful. Instead of performing single shifts k times to achieve a multiplication by $2^{\pm k}$ a mixture of whole digit shifting and partial digit shifting is employed.

4.5.1 Multiplication by Power of Two

Algorithm **mp_mul_2d**.
Input. One mp_int a and an integer b
Output. $c \leftarrow a \cdot 2^b$.

1. $c \leftarrow a$. (mp_copy)
2. If $c.alloc < c.used + \lfloor b/lg(\beta) \rfloor + 2$ then grow c accordingly.
3. If the reallocation failed return(MP_MEM).
4. If $b \geq lg(\beta)$ then
   4.1 $c \leftarrow c \cdot \beta^{\lfloor b/lg(\beta) \rfloor}$ (mp_lshd).
   4.2 If step 4.1 failed return(MP_MEM).
5. $d \leftarrow b \pmod{lg(\beta)}$
6. If $d \neq 0$ then do
   6.1 $mask \leftarrow 2^d$
   6.2 $r \leftarrow 0$
   6.3 for n from 0 to $c.used - 1$ do
      6.3.1 $rr \leftarrow c_n >> (lg(\beta) - d) \pmod{mask}$
      6.3.2 $c_n \leftarrow (c_n << d) + r \pmod{\beta}$
      6.3.3 $r \leftarrow rr$
   6.4 If $r > 0$ then do
      6.4.1 $c_{c.used} \leftarrow r$
      6.4.2 $c.used \leftarrow c.used + 1$
7. Return(MP_OKAY).

Figure 4.12: Algorithm mp_mul_2d

Algorithm mp_mul_2d. This algorithm multiplies a by 2^b and stores the result in c. The algorithm uses algorithm mp_lshd and a derivative of algorithm mp_mul_2 to quickly compute the product.

First the algorithm will multiply a by $x^{\lfloor b/lg(\beta) \rfloor}$ which will ensure that the remainder multiplicand is less than β. For example, if $b = 37$ and $\beta = 2^{28}$ then this step will multiply by x leaving a multiplication by $2^{37-28} = 2^9$ left.

After the digits have been shifted appropriately at most $lg(\beta) - 1$ shifts are left to perform. Step 5 calculates the number of remaining shifts required. If it is non-zero a modified shift loop is used to calculate the remaining product. Essentially the loop is a generic version of algorithm mp_mul_2 designed to handle any shift count in the range $1 \leq x < lg(\beta)$. The $mask$ variable is used to extract the upper d bits to form the carry for the next iteration.

This algorithm is loosely measured as a $O(2n)$ algorithm which means that if the input is n-digits that it takes $2n$ "time" to complete. It is possible to optimize this algorithm down to a $O(n)$ algorithm at a cost of making the

algorithm slightly harder to follow.

File: bn_mp_mul_2d.c

```
001   #ifdef BN_MP_MUL_2D_C
...
```

The shifting is performed in–place which means the first step (line 26) is to copy the input to the destination. We avoid calling mp_copy() by making sure the mp_ints are different. The destination then has to be grown (line 33) to accomodate the result.

If the shift count b is larger than $lg(\beta)$ then a call to mp_lshd() is used to handle all of the multiples of $lg(\beta)$. Leaving only a remaining shift of $lg(\beta) - 1$ or fewer bits left. Inside the actual shift loop (lines 45 to 77) we make use of pre–computed values $shift$ and $mask$. These are used to extract the carry bit(s) to pass into the next iteration of the loop. The r and rr variables form a chain between consecutive iterations to propagate the carry.

4.5.2 Division by Power of Two

Algorithm **mp_div_2d**.
Input. One mp_int a and an integer b
Output. $c \leftarrow \lfloor a/2^b \rfloor, d \leftarrow a \pmod{2^b}$.

1. If $b \leq 0$ then do
   1.1 $c \leftarrow a$ (*mp_copy*)
   1.2 $d \leftarrow 0$ (*mp_zero*)
   1.3 Return(*MP_OKAY*).
2. $c \leftarrow a$
3. $d \leftarrow a \pmod{2^b}$ (*mp_mod_2d*)
4. If $b \geq lg(\beta)$ then do
   4.1 $c \leftarrow \lfloor c/\beta^{\lfloor b/lg(\beta) \rfloor} \rfloor$ (*mp_rshd*).
5. $k \leftarrow b \pmod{lg(\beta)}$
6. If $k \neq 0$ then do
   6.1 $mask \leftarrow 2^k$
   6.2 $r \leftarrow 0$
   6.3 for n from $c.used - 1$ to 0 do
      6.3.1 $rr \leftarrow c_n \pmod{mask}$
      6.3.2 $c_n \leftarrow (c_n >> k) + (r << (lg(\beta) - k))$
      6.3.3 $r \leftarrow rr$
7. Clamp excess digits of c. (*mp_clamp*)
8. Return(*MP_OKAY*).

Figure 4.13: Algorithm mp_div_2d

Algorithm mp_div_2d. This algorithm will divide an input a by 2^b and produce the quotient and remainder. The algorithm is designed much like algorithm mp_mul_2d by first using whole digit shifts then single precision shifts. This algorithm will also produce the remainder of the division by using algorithm mp_mod_2d.

File: bn_mp_div_2d.c

```
001   #ifdef BN_MP_DIV_2D_C
...
```

The implementation of algorithm mp_div_2d is slightly different than the algorithm specifies. The remainder d may be optionally ignored by passing **NULL** as the pointer to the mp_int variable. The temporary mp_int variable t is used to hold the result of the remainder operation until the end. This allows d and a to represent the same mp_int without modifying a before the quotient is obtained.

The remainder of the source code is essentially the same as the source code for mp_mul_2d. The only significant difference is the direction of the shifts.

4.5.3 Remainder of Division by Power of Two

The last algorithm in the series of polynomial basis power of two algorithms is calculating the remainder of division by 2^b. This algorithm benefits from the fact that in twos complement arithmetic $a \pmod{2^b}$ is the same as a AND $2^b - 1$.

Algorithm **mp_mod_2d**.
Input. One mp_int a and an integer b
Output. $c \leftarrow a \pmod{2^b}$.

1. If $b \leq 0$ then do
   1.1 $c \leftarrow 0$ (mp_zero)
   1.2 Return(MP_OKAY).
2. If $b > a.used \cdot lg(\beta)$ then do
   2.1 $c \leftarrow a$ (mp_copy)
   2.2 Return the result of step 2.1.
3. $c \leftarrow a$
4. If step 3 failed return(MP_MEM).
5. for n from $\lceil b/lg(\beta) \rceil$ to $c.used$ do
   5.1 $c_n \leftarrow 0$
6. $k \leftarrow b \pmod{lg(\beta)}$
7. $c_{\lfloor b/lg(\beta) \rfloor} \leftarrow c_{\lfloor b/lg(\beta) \rfloor} \pmod{2^k}$.
8. Clamp excess digits of c. (mp_clamp)
9. Return(MP_OKAY).

Figure 4.14: Algorithm mp_mod_2d

Algorithm mp_mod_2d. This algorithm will quickly calculate the value of $a \pmod{2^b}$. First if b is less than or equal to zero the result is set to zero. If b is greater than the number of bits in a then it simply copies a to c and returns. Otherwise, a is copied to b, leading digits are removed and the remaining leading digit is trimed to the exact bit count.

File: bn_mp_mod_2d.c

```
001   #ifdef BN_MP_MOD_2D_C
...
```

We first avoid cases of $b \leq 0$ by simply mp_zero()'ing the destination in such cases. Next if 2^b is larger than the input we just mp_copy() the input and return right away. After this point we know we must actually perform some work to produce the remainder.

Recalling that reducing modulo 2^k and a binary "and" with $2^k - 1$ are numerically equivalent we can quickly reduce the number. First we zero any digits above the last digit in 2^b (line 38). Next we reduce the leading digit of both (line 42) and then mp_clamp().

Exercises

[3] Devise an algorithm that performs $a \cdot 2^b$ for generic values of b in $O(n)$ time.

[3] Devise an efficient algorithm to multiply by small low hamming weight values such as 3, 5 and 9. Extend it to handle all values upto 64 with a hamming weight less than three.

[2] Modify the preceding algorithm to handle values of the form $2^k - 1$ as well.

[3] Using only algorithms mp_mul_2, mp_div_2 and mp_add create an algorithm to multiply two integers in roughly $O(2n^2)$ time for any n-bit input. Note that the time of addition is ignored in the calculation.

[5] Improve the previous algorithm to have a working time of at most $O\left(2^{(k-1)}n + \left(\frac{2n^2}{k}\right)\right)$ for an appropriate choice of k. Again ignore the cost of addition.

[2] Devise a chart to find optimal values of k for the previous problem for $n = 64 \ldots 1024$ in steps of 64.

[2] Using only algorithms mp_abs and mp_sub devise another method for calculating the result of a signed comparison.

Chapter 5

Multiplication and Squaring

5.1 The Multipliers

For most number theoretic problems including certain public key cryptographic algorithms, the "multipliers" form the most important subset of algorithms of any multiple precision integer package. The set of multiplier algorithms include integer multiplication, squaring and modular reduction where in each of the algorithms single precision multiplication is the dominant operation performed. This chapter will discuss integer multiplication and squaring, leaving modular reductions for the subsequent chapter.

The importance of the multiplier algorithms is for the most part driven by the fact that certain popular public key algorithms are based on modular exponentiation, that is computing $d \equiv a^b \pmod{c}$ for some arbitrary choice of a, b, c and d. During a modular exponentiation the majority[1] of the processor time is spent performing single precision multiplications.

For centuries general purpose multiplication has required a lengthly $O(n^2)$ process, whereby each digit of one multiplicand has to be multiplied against every digit of the other multiplicand. Traditional long-hand multiplication is based on this process; while the techniques can differ the overall algorithm used is essentially the same. Only "recently" have faster algorithms been studied. First Karatsuba multiplication was discovered in 1962. This algorithm can

[1] Roughly speaking a modular exponentiation will spend about 40% of the time performing modular reductions, 35% of the time performing squaring and 25% of the time performing multiplications.

multiply two numbers with considerably fewer single precision multiplications when compared to the long-hand approach. This technique led to the discovery of polynomial basis algorithms (*good reference?*) and subquently Fourier Transform based solutions.

5.2 Multiplication

5.2.1 The Baseline Multiplication

Computing the product of two integers in software can be achieved using a trivial adaptation of the standard $O(n^2)$ long-hand multiplication algorithm that school children are taught. The algorithm is considered an $O(n^2)$ algorithm since for two n-digit inputs n^2 single precision multiplications are required. More specifically for a m and n digit input $m \cdot n$ single precision multiplications are required. To simplify most discussions, it will be assumed that the inputs have comparable number of digits.

The "baseline multiplication" algorithm is designed to act as the "catch-all" algorithm, only to be used when the faster algorithms cannot be used. This algorithm does not use any particularly interesting optimizations and should ideally be avoided if possible. One important facet of this algorithm, is that it has been modified to only produce a certain amount of output digits as resolution. The importance of this modification will become evident during the discussion of Barrett modular reduction. Recall that for a n and m digit input the product will be at most $n + m$ digits. Therefore, this algorithm can be reduced to a full multiplier by having it produce $n + m$ digits of the product.

Recall from sub-section 4.2.2 the definition of γ as the number of bits in the type **mp_digit**. We shall now extend the variable set to include α which shall represent the number of bits in the type **mp_word**. This implies that $2^\alpha > 2 \cdot \beta^2$. The constant $\delta = 2^{\alpha - 2lg(\beta)}$ will represent the maximal weight of any column in a product (*see sub-section 5.2.2 for more information*).

Algorithm **s_mp_mul_digs**.
Input. mp_int a, mp_int b and an integer $digs$
Output. $c \leftarrow |a| \cdot |b| \pmod{\beta^{digs}}$.

1. If $\min(a.used, b.used) < \delta$ then do
   1.1 Calculate $c = |a| \cdot |b|$ by the Comba method (*see algorithm 5.5*).
   1.2 Return the result of step 1.1

Allocate and initialize a temporary mp_int.
2. Init t to be of size $digs$
3. If step 2 failed return(MP_MEM).
4. $t.used \leftarrow digs$

Compute the product.
5. for ix from 0 to $a.used - 1$ do
   5.1 $u \leftarrow 0$
   5.2 $pb \leftarrow \min(b.used, digs - ix)$
   5.3 If $pb < 1$ then goto step 6.
   5.4 for iy from 0 to $pb - 1$ do
      5.4.1 $\hat{r} \leftarrow t_{iy+ix} + a_{ix} \cdot b_{iy} + u$
      5.4.2 $t_{iy+ix} \leftarrow \hat{r} \pmod{\beta}$
      5.4.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
   5.5 if $ix + pb < digs$ then do
      5.5.1 $t_{ix+pb} \leftarrow u$
6. Clamp excess digits of t.
7. Swap c with t
8. Clear t
9. Return(MP_OKAY).

Figure 5.1: Algorithm s_mp_mul_digs

Algorithm s_mp_mul_digs. This algorithm computes the unsigned product of two inputs a and b, limited to an output precision of $digs$ digits. While it may seem a bit awkward to modify the function from its simple $O(n^2)$ description, the usefulness of partial multipliers will arise in a subsequent algorithm. The algorithm is loosely based on algorithm 14.12 from [2, pp. 595] and is similar to Algorithm M of Knuth [1, pp. 268]. Algorithm s_mp_mul_digs differs from these cited references since it can produce a variable output precision regardless of the precision of the inputs.

The first thing this algorithm checks for is whether a Comba multiplier can

be used instead. If the minimum digit count of either input is less than δ, then the Comba method may be used instead. After the Comba method is ruled out, the baseline algorithm begins. A temporary mp_int variable t is used to hold the intermediate result of the product. This allows the algorithm to be used to compute products when either $a = c$ or $b = c$ without overwriting the inputs.

All of step 5 is the infamous $O(n^2)$ multiplication loop slightly modified to only produce upto *digs* digits of output. The *pb* variable is given the count of digits to read from b inside the nested loop. If $pb \leq 1$ then no more output digits can be produced and the algorithm will exit the loop. The best way to think of the loops are as a series of $pb \times 1$ multiplications. That is, in each pass of the innermost loop a_{ix} is multiplied against b and the result is added (*with an appropriate shift*) to t.

For example, consider multiplying 576 by 241. That is equivalent to computing $10^0(1)(576) + 10^1(4)(576) + 10^2(2)(576)$ which is best visualized in the following table.

			5	7	6	
$\times$			2	4	1	
			5	7	6	$10^0(1)(576)$
	2	3	6	1	6	$10^1(4)(576) + 10^0(1)(576)$
1	3	8	8	1	6	$10^2(2)(576) + 10^1(4)(576) + 10^0(1)(576)$

Figure 5.2: Long-Hand Multiplication Diagram

Each row of the product is added to the result after being shifted to the left (*multiplied by a power of the radix*) by the appropriate count. That is in pass ix of the inner loop the product is added starting at the ix'th digit of the reult.

Step 5.4.1 introduces the hat symbol (*e.g.* $\hat{r}$) which represents a double precision variable. The multiplication on that step is assumed to be a double wide output single precision multiplication. That is, two single precision variables are multiplied to produce a double precision result. The step is somewhat optimized from a long-hand multiplication algorithm because the carry from the addition in step 5.4.1 is propagated through the nested loop. If the carry was not propagated immediately it would overflow the single precision digit t_{ix+iy} and the result would be lost.

At step 5.5 the nested loop is finished and any carry that was left over should be forwarded. The carry does not have to be added to the $ix + pb$'th digit since

that digit is assumed to be zero at this point. However, if $ix + pb \geq digs$ the carry is not set as it would make the result exceed the precision requested.

File: bn_s_mp_mul_digs.c

```
001   #ifdef BN_S_MP_MUL_DIGS_C
...
```

First we determine (line 32) if the Comba method can be used first since it's faster. The conditions for sing the Comba routine are that $\min(a.used, b.used) < \delta$ and the number of digits of output is less than **MP_WARRAY**. This new constant is used to control the stack usage in the Comba routines. By default it is set to δ but can be reduced when memory is at a premium.

If we cannot use the Comba method we proceed to setup the baseline routine. We allocate the the destination mp_int t (line 38) to the exact size of the output to avoid further re–allocations. At this point we now begin the $O(n^2)$ loop.

This implementation of multiplication has the caveat that it can be trimmed to only produce a variable number of digits as output. In each iteration of the outer loop the *pb* variable is set (line 50) to the maximum number of inner loop iterations.

Inside the inner loop we calculate $\hat{r}$ as the mp_word product of the two mp_digits and the addition of the carry from the previous iteration. A particularly important observation is that most modern optimizing C compilers (GCC for instance) can recognize that a $N \times N \rightarrow 2N$ multiplication is all that is required for the product. In x86 terms for example, this means using the MUL instruction.

Each digit of the product is stored in turn (line 70) and the carry propagated (line 73) to the next iteration.

5.2.2 Faster Multiplication by the "Comba" Method

One of the huge drawbacks of the "baseline" algorithms is that at the $O(n^2)$ level the carry must be computed and propagated upwards. This makes the nested loop very sequential and hard to unroll and implement in parallel. The "Comba" [4] method is named after little known (*in cryptographic venues*) Paul G. Comba who described a method of implementing fast multipliers that do not require nested carry fixup operations. As an interesting aside it seems that Paul Barrett describes a similar technique in his 1986 paper [7] written five years before.

At the heart of the Comba technique is once again the long-hand algorithm. Except in this case a slight twist is placed on how the columns of the result are

produced. In the standard long-hand algorithm rows of products are produced then added together to form the final result. In the baseline algorithm the columns are added together after each iteration to get the result instantaneously.

In the Comba algorithm the columns of the result are produced entirely independently of each other. That is at the $O(n^2)$ level a simple multiplication and addition step is performed. The carries of the columns are propagated after the nested loop to reduce the amount of work requiored. Succintly the first step of the algorithm is to compute the product vector $\vec{x}$ as follows.

$$\vec{x}_n = \sum_{i+j=n} a_i b_j, \forall n \in \{0, 1, 2, \ldots, i+j\} \tag{5.1}$$

Where $\vec{x}_n$ is the $n'th$ column of the output vector. Consider the following example which computes the vector $\vec{x}$ for the multiplication of 576 and 241.

		5	7	6	First Input
$\times$		2	4	1	Second Input
		$1 \cdot 5 = 5$	$1 \cdot 7 = 7$	$1 \cdot 6 = 6$	First pass
	$4 \cdot 5 = 20$	$4 \cdot 7 + 5 = 33$	$4 \cdot 6 + 7 = 31$	6	Second pass
$2 \cdot 5 = 10$	$2 \cdot 7 + 20 = 34$	$2 \cdot 6 + 33 = 45$	31	6	Third pass
10	34	45	31	6	Final Result

Figure 5.3: Comba Multiplication Diagram

At this point the vector $x = \langle 10, 34, 45, 31, 6 \rangle$ is the result of the first step of the Comba multipler. Now the columns must be fixed by propagating the carry upwards. The resultant vector will have one extra dimension over the input vector which is congruent to adding a leading zero digit.

Algorithm **Comba Fixup**.
Input. Vector $\vec{x}$ of dimension k
Output. Vector $\vec{x}$ such that the carries have been propagated.

1. for n from 0 to $k - 1$ do
   1.1 $\vec{x}_{n+1} \leftarrow \vec{x}_{n+1} + \lfloor \vec{x}_n / \beta \rfloor$
   1.2 $\vec{x}_n \leftarrow \vec{x}_n \pmod{\beta}$
2. Return($\vec{x}$).

Figure 5.4: Algorithm Comba Fixup

With that algorithm and $k = 5$ and $\beta = 10$ the following vector is produced $\vec{x} = \langle 1, 3, 8, 8, 1, 6 \rangle$. In this case $241 \cdot 576$ is in fact 138816 and the procedure succeeded. If the algorithm is correct and as will be demonstrated shortly more efficient than the baseline algorithm why not simply always use this algorithm?

Column Weight.

At the nested $O(n^2)$ level the Comba method adds the product of two single precision variables to each column of the output independently. A serious obstacle is if the carry is lost, due to lack of precision before the algorithm has a chance to fix the carries. For example, in the multiplication of two three-digit numbers the third column of output will be the sum of three single precision multiplications. If the precision of the accumulator for the output digits is less then $3 \cdot (\beta - 1)^2$ then an overflow can occur and the carry information will be lost.

For any m and n digit inputs the maximum weight of any column is $\min(m, n)$ which is fairly obvious.

The maximum number of terms in any column of a product is known as the "column weight" and strictly governs when the algorithm can be used. Recall from earlier that a double precision type has α bits of resolution and a single precision digit has $lg(\beta)$ bits of precision. Given these two quantities we must not violate the following

$$k \cdot (\beta - 1)^2 < 2^\alpha \tag{5.2}$$

Which reduces to

$$k \cdot \left(\beta^2 - 2\beta + 1\right) < 2^\alpha \tag{5.3}$$

Let $\rho = lg(\beta)$ represent the number of bits in a single precision digit. By further re-arrangement of the equation the final solution is found.

$$k < \frac{2^\alpha}{(2^{2\rho} - 2^{\rho+1} + 1)} \tag{5.4}$$

The defaults for LibTomMath are $\beta = 2^{28}$ and $\alpha = 2^{64}$ which means that k is bounded by $k < 257$. In this configuration the smaller input may not have more than 256 digits if the Comba method is to be used. This is quite satisfactory for most applications since 256 digits would allow for numbers in the range of $0 \le x < 2^{7168}$ which, is much larger than most public key cryptographic algorithms require.

Algorithm **fast_s_mp_mul_digs**.
Input. mp_int a, mp_int b and an integer $digs$
Output. $c \leftarrow |a| \cdot |b| \pmod{\beta^{digs}}$.

Place an array of **MP_WARRAY** single precision digits named W on the stack.
1. If $c.alloc < digs$ then grow c to $digs$ digits. (mp_grow)
2. If step 1 failed return(MP_MEM).

3. $pa \leftarrow \text{MIN}(digs, a.used + b.used)$

4. $_\hat{W} \leftarrow 0$
5. for ix from 0 to $pa - 1$ do
   5.1 $ty \leftarrow \text{MIN}(b.used - 1, ix)$
   5.2 $tx \leftarrow ix - ty$
   5.3 $iy \leftarrow \text{MIN}(a.used - tx, ty + 1)$
   5.4 for iz from 0 to $iy - 1$ do
      5.4.1 $_\hat{W} \leftarrow _\hat{W} + a_{tx+iy}b_{ty-iy}$
   5.5 $W_{ix} \leftarrow _\hat{W} \pmod{\beta}$
   5.6 $_\hat{W} \leftarrow \lfloor _\hat{W}/\beta \rfloor$

6. $oldused \leftarrow c.used$
7. $c.used \leftarrow digs$
8. for ix from 0 to pa do
   8.1 $c_{ix} \leftarrow W_{ix}$
9. for ix from $pa + 1$ to $oldused - 1$ do
   9.1 $c_{ix} \leftarrow 0$

10. Clamp c.
11. Return MP_OKAY.

Figure 5.5: Algorithm fast_s_mp_mul_digs

Algorithm fast_s_mp_mul_digs. This algorithm performs the unsigned multiplication of a and b using the Comba method limited to $digs$ digits of precision.

The outer loop of this algorithm is more complicated than that of the baseline multiplier. This is because on the inside of the loop we want to produce one column per pass. This allows the accumulator $_\hat{W}$ to be placed in CPU registers and reduce the memory bandwidth to two **mp_digit** reads per iteration.

The ty variable is set to the minimum count of ix or the number of digits in

b. That way if a has more digits than b this will be limited to $b.used - 1$. The tx variable is set to the to the distance past $b.used$ the variable ix is. This is used for the immediately subsequent statement where we find iy.

The variable iy is the minimum digits we can read from either a or b before running out. Computing one column at a time means we have to scan one integer upwards and the other downwards. a starts at tx and b starts at ty. In each pass we are producing the ix'th output column and we note that $tx + ty = ix$. As we move tx upwards we have to move ty downards so the equality remains valid. The iy variable is the number of iterations until $tx \geq a.used$ or $ty < 0$ occurs.

After every inner pass we store the lower half of the accumulator into W_{ix} and then propagate the carry of the accumulator into the next round by dividing $_\hat{W}$ by β.

To measure the benefits of the Comba method over the baseline method consider the number of operations that are required. If the cost in terms of time of a multiply and addition is p and the cost of a carry propagation is q then a baseline multiplication would require $O\left((p+q)n^2\right)$ time to multiply two n-digit numbers. The Comba method requires only $O(pn^2 + qn)$ time, however in practice, the speed increase is actually much more. With $O(n)$ space the algorithm can be reduced to $O(pn + qn)$ time by implementing the n multiply and addition operations in the nested loop in parallel.

File: bn_fast_s_mp_mul_digs.c

```
001   #ifdef BN_FAST_S_MP_MUL_DIGS_C
...
```

As per the pseudo–code we first calculate pa (line 45) as the number of digits to output. Next we begin the outer loop to produce the individual columns of the product. We use the two aliases $tmpx$ and $tmpy$ (lines 59, 60) to point inside the two multiplicands quickly.

The inner loop (lines 68 to 71) of this implementation is where the tradeoff come into play. Originally this comba implementation was "row–major" which means it adds to each of the columns in each pass. After the outer loop it would then fix the carries. This was very fast except it had an annoying drawback. You had to read a mp_word and two mp_digits and write one mp_word per iteration. On processors such as the Athlon XP and P4 this did not matter much since the cache bandwidth is very high and it can keep the ALU fed with data. It did, however, matter on older and embedded cpus where cache is often slower and also often doesn't exist. This new algorithm only performs two reads

per iteration under the assumption that the compiler has aliased $_\hat{W}$ to a CPU register.

After the inner loop we store the current accumulator in W and shift $_\hat{W}$ (lines 74, 77) to forward it as a carry for the next pass. After the outer loop we use the final carry (line 89) as the last digit of the product.

5.2.3 Polynomial Basis Multiplication

To break the $O(n^2)$ barrier in multiplication requires a completely different look at integer multiplication. In the following algorithms the use of polynomial basis representation for two integers a and b as $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$ respectively, is required. In this system both $f(x)$ and $g(x)$ have $n+1$ terms and are of the n'th degree.

The product $a \cdot b \equiv f(x)g(x)$ is the polynomial $W(x) = \sum_{i=0}^{2n} w_i x^i$. The coefficients w_i will directly yield the desired product when β is substituted for x. The direct solution to solve for the $2n+1$ coefficients requires $O(n^2)$ time and would in practice be slower than the Comba technique.

However, numerical analysis theory indicates that only $2n+1$ distinct points in $W(x)$ are required to determine the values of the $2n+1$ unknown coefficients. This means by finding $\zeta_y = W(y)$ for $2n+1$ small values of y the coefficients of $W(x)$ can be found with Gaussian elimination. This technique is also occasionally refered to as the *interpolation technique* (*references please...*) since in effect an interpolation based on $2n+1$ points will yield a polynomial equivalent to $W(x)$.

The coefficients of the polynomial $W(x)$ are unknown which makes finding $W(y)$ for any value of y impossible. However, since $W(x) = f(x)g(x)$ the equivalent $\zeta_y = f(y)g(y)$ can be used in its place. The benefit of this technique stems from the fact that $f(y)$ and $g(y)$ are much smaller than either a or b respectively. As a result finding the $2n+1$ relations required by multiplying $f(y)g(y)$ involves multiplying integers that are much smaller than either of the inputs.

When picking points to gather relations there are always three obvious points to choose, $y = 0, 1$ and ∞. The ζ_0 term is simply the product $W(0) = w_0 = a_0 \cdot b_0$. The ζ_1 term is the product $W(1) = (\sum_{i=0}^{n} a_i)(\sum_{i=0}^{n} b_i)$. The third point ζ_∞ is less obvious but rather simple to explain. The $2n+1$'th coefficient of $W(x)$ is numerically equivalent to the most significant column in an integer multiplication. The point at ∞ is used symbolically to represent the most significant column, that is $W(\infty) = w_{2n} = a_n b_n$. Note that the points at $y = 0$ and ∞ yield the coefficients w_0 and w_{2n} directly.

Split into n Parts	**Exponent**	**Notes**
2	1.584962501	This is Karatsuba Multiplication.
3	1.464973520	This is Toom-Cook Multiplication.
4	1.403677461	
5	1.365212389	
10	1.278753601	
100	1.149426538	
1000	1.100270931	
10000	1.075252070	

Figure 5.6: Asymptotic Running Time of Polynomial Basis Multiplication

If more points are required they should be of small values and powers of two such as 2^q and the related *mirror points* $(2^q)^{2n} \cdot \zeta_{2^{-q}}$ for small values of q. The term "mirror point" stems from the fact that $(2^q)^{2n} \cdot \zeta_{2^{-q}}$ can be calculated in the exact opposite fashion as ζ_{2^q}. For example, when $n = 2$ and $q = 1$ then following two equations are equivalent to the point ζ_2 and its mirror.

$$\begin{aligned} \zeta_2 = f(2)g(2) &= (4a_2 + 2a_1 + a_0)(4b_2 + 2b_1 + b_0) \\ 16 \cdot \zeta_{\frac{1}{2}} = 4f(\frac{1}{2}) \cdot 4g(\frac{1}{2}) &= (a_2 + 2a_1 + 4a_0)(b_2 + 2b_1 + 4b_0) \end{aligned} \tag{5.5}$$

Using such points will allow the values of $f(y)$ and $g(y)$ to be independently calculated using only left shifts. For example, when $n = 2$ the polynomial $f(2^q)$ is equal to $2^q((2^q a_2) + a_1) + a_0$. This technique of polynomial representation is known as Horner's method.

As a general rule of the algorithm when the inputs are split into n parts each there are $2n - 1$ multiplications. Each multiplication is of multiplicands that have n times fewer digits than the inputs. The asymptotic running time of this algorithm is $O\left(k^{lg_n(2n-1)}\right)$ for k digit inputs (*assuming they have the same number of digits*). Figure 5.6 summarizes the exponents for various values of n.

At first it may seem like a good idea to choose $n = 1000$ since the exponent is approximately 1.1. However, the overhead of solving for the 2001 terms of $W(x)$ will certainly consume any savings the algorithm could offer for all but exceedingly large numbers.

Cutoff Point

The polynomial basis multiplication algorithms all require fewer single precision multiplications than a straight Comba approach. However, the algorithms incur an overhead (*at the $O(n)$ work level*) since they require a system of equations to be solved. This makes the polynomial basis approach more costly to use with small inputs.

Let m represent the number of digits in the multiplicands (*assume both multiplicands have the same number of digits*). There exists a point y such that when $m < y$ the polynomial basis algorithms are more costly than Comba, when $m = y$ they are roughly the same cost and when $m > y$ the Comba methods are slower than the polynomial basis algorithms.

The exact location of y depends on several key architectural elements of the computer platform in question.

1. The ratio of clock cycles for single precision multiplication versus other simpler operations such as addition, shifting, etc. For example on the AMD Athlon the ratio is roughly $17 : 1$ while on the Intel P4 it is $29 : 1$. The higher the ratio in favour of multiplication the lower the cutoff point y will be.

2. The complexity of the linear system of equations (*for the coefficients of $W(x)$*) is. Generally speaking as the number of splits grows the complexity grows substantially. Ideally solving the system will only involve addition, subtraction and shifting of integers. This directly reflects on the ratio previous mentioned.

3. To a lesser extent memory bandwidth and function call overheads. Provided the values are in the processor cache this is less of an influence over the cutoff point.

A clean cutoff point separation occurs when a point y is found such that all of the cutoff point conditions are met. For example, if the point is too low then there will be values of m such that $m > y$ and the Comba method is still faster. Finding the cutoff points is fairly simple when a high resolution timer is available.

5.2.4 Karatsuba Multiplication

Karatsuba [5] multiplication when originally proposed in 1962 was among the first set of algorithms to break the $O(n^2)$ barrier for general purpose mul-

tiplication. Given two polynomial basis representations $f(x) = ax + b$ and $g(x) = cx + d$, Karatsuba proved with light algebra [6] that the following polynomial is equivalent to multiplication of the two integers the polynomials represent.

$$f(x) \cdot g(x) = acx^2 + ((a + b)(c + d) - (ac + bd))x + bd \tag{5.6}$$

Using the observation that ac and bd could be re-used only three half sized multiplications would be required to produce the product. Applying this algorithm recursively, the work factor becomes $O(n^{lg(3)})$ which is substantially better than the work factor $O(n^2)$ of the Comba technique. It turns out what Karatsuba did not know or at least did not publish was that this is simply polynomial basis multiplication with the points ζ_0, ζ_∞ and ζ_1. Consider the resultant system of equations.

$$\begin{array}{rcccccc} \zeta_0 & = & & & & & w_0 \\ \zeta_1 & = & w_2 & + & w_1 & + & w_0 \\ \zeta_\infty & = & w_2 & & & & \end{array}$$

By adding the first and last equation to the equation in the middle the term w_1 can be isolated and all three coefficients solved for. The simplicity of this system of equations has made Karatsuba fairly popular. In fact the cutoff point is often fairly low[2] making it an ideal algorithm to speed up certain public key cryptosystems such as RSA and Diffie-Hellman.

[2]With LibTomMath 0.18 it is 70 and 109 digits for the Intel P4 and AMD Athlon respectively.

Algorithm **mp_karatsuba_mul**.
Input. mp_int a and mp_int b
Output. $c \leftarrow |a| \cdot |b|$

1. Init the following mp_int variables: $x0$, $x1$, $y0$, $y1$, $t1$, $x0y0$, $x1y1$.
2. If step 2 failed then return(MP_MEM).

Split the input. e.g. $a = x1 \cdot \beta^B + x0$
3. $B \leftarrow \min(a.used, b.used)/2$
4. $x0 \leftarrow a \pmod{\beta^B}$ (mp_mod_2d)
5. $y0 \leftarrow b \pmod{\beta^B}$
6. $x1 \leftarrow \lfloor a/\beta^B \rfloor$ (mp_rshd)
7. $y1 \leftarrow \lfloor b/\beta^B \rfloor$

Calculate the three products.
8. $x0y0 \leftarrow x0 \cdot y0$ (mp_mul)
9. $x1y1 \leftarrow x1 \cdot y1$
10. $t1 \leftarrow x1 + x0$ (mp_add)
11. $x0 \leftarrow y1 + y0$
12. $t1 \leftarrow t1 \cdot x0$

Calculate the middle term.
13. $x0 \leftarrow x0y0 + x1y1$
14. $t1 \leftarrow t1 - x0$ (s_mp_sub)

Calculate the final product.
15. $t1 \leftarrow t1 \cdot \beta^B$ (mp_lshd)
16. $x1y1 \leftarrow x1y1 \cdot \beta^{2B}$
17. $t1 \leftarrow x0y0 + t1$
18. $c \leftarrow t1 + x1y1$
19. Clear all of the temporary variables.
20. Return(MP_OKAY).

Figure 5.7: Algorithm mp_karatsuba_mul

Algorithm mp_karatsuba_mul. This algorithm computes the unsigned product of two inputs using the Karatsuba multiplication algorithm. It is loosely based on the description from Knuth [1, pp. 294-295].

In order to split the two inputs into their respective halves, a suitable *radix point* must be chosen. The radix point chosen must be used for both of the inputs meaning that it must be smaller than the smallest input. Step 3 chooses

the radix point B as half of the smallest input **used** count. After the radix point is chosen the inputs are split into lower and upper halves. Step 4 and 5 compute the lower halves. Step 6 and 7 computer the upper halves.

After the halves have been computed the three intermediate half-size products must be computed. Step 8 and 9 compute the trivial products $x0 \cdot y0$ and $x1 \cdot y1$. The mp_int $x0$ is used as a temporary variable after $x1 + x0$ has been computed. By using $x0$ instead of an additional temporary variable, the algorithm can avoid an addition memory allocation operation.

The remaining steps 13 through 18 compute the Karatsuba polynomial through a variety of digit shifting and addition operations.

File: bn_mp_karatsuba_mul.c

```
001   #ifdef BN_MP_KARATSUBA_MUL_C
...
```

The new coding element in this routine, not seen in previous routines, is the usage of goto statements. The conventional wisdom is that goto statements should be avoided. This is generally true, however when every single function call can fail, it makes sense to handle error recovery with a single piece of code. Lines 61 to 76 handle initializing all of the temporary variables required. Note how each of the if statements goes to a different label in case of failure. This allows the routine to correctly free only the temporaries that have been successfully allocated so far.

The temporary variables are all initialized using the mp_init_size routine since they are expected to be large. This saves the additional reallocation that would have been necessary. Also $x0$, $x1$, $y0$ and $y1$ have to be able to hold at least their respective number of digits for the next section of code.

The first algebraic portion of the algorithm is to split the two inputs into their halves. However, instead of using mp_mod_2d and mp_rshd to extract the halves, the respective code has been placed inline within the body of the function. To initialize the halves, the **used** and **sign** members are copied first. The first for loop on line 99 copies the lower halves. Since they are both the same magnitude it is simpler to calculate both lower halves in a single loop. The for loop on lines 104 and 104 calculate the upper halves $x1$ and $y1$ respectively.

By inlining the calculation of the halves, the Karatsuba multiplier has a slightly lower overhead and can be used for smaller magnitude inputs.

When line 148 is reached, the algorithm has completed succesfully. The "error status" variable *err* is set to **MP_OKAY** so that the same code that handles errors can be used to clear the temporary variables and return.

5.2.5 Toom-Cook 3-Way Multiplication

Toom-Cook 3-Way [22] multiplication is essentially the polynomial basis algorithm for $n = 2$ except that the points are chosen such that ζ is easy to compute and the resulting system of equations easy to reduce. Here, the points ζ_0, $16 \cdot \zeta_{\frac{1}{2}}$, ζ_1, ζ_2 and ζ_∞ make up the five required points to solve for the coefficients of the $W(x)$.

With the five relations that Toom-Cook specifies, the following system of equations is formed.

$$\begin{aligned}
\zeta_0 &= 0w_4 + 0w_3 + 0w_2 + 0w_1 + 1w_0 \\
16 \cdot \zeta_{\frac{1}{2}} &= 1w_4 + 2w_3 + 4w_2 + 8w_1 + 16w_0 \\
\zeta_1 &= 1w_4 + 1w_3 + 1w_2 + 1w_1 + 1w_0 \\
\zeta_2 &= 16w_4 + 8w_3 + 4w_2 + 2w_1 + 1w_0 \\
\zeta_\infty &= 1w_4 + 0w_3 + 0w_2 + 0w_1 + 0w_0
\end{aligned}$$

A trivial solution to this matrix requires 12 subtractions, two multiplications by a small power of two, two divisions by a small power of two, two divisions by three and one multiplication by three. All of these 19 sub-operations require less than quadratic time, meaning that the algorithm can be faster than a baseline multiplication. However, the greater complexity of this algorithm places the cutoff point (**TOOM_MUL_CUTOFF**) where Toom-Cook becomes more efficient much higher than the Karatsuba cutoff point.

Algorithm **mp_toom_mul**.
Input. mp_int a and mp_int b
Output. $c \leftarrow a \cdot b$

Split a and b into three pieces. E.g. $a = a_2\beta^{2k} + a_1\beta^k + a_0$
1. $k \leftarrow \lfloor \min(a.used, b.used)/3 \rfloor$
2. $a_0 \leftarrow a \pmod{\beta^k}$
3. $a_1 \leftarrow \lfloor a/\beta^k \rfloor$, $a_1 \leftarrow a_1 \pmod{\beta^k}$
4. $a_2 \leftarrow \lfloor a/\beta^{2k} \rfloor$, $a_2 \leftarrow a_2 \pmod{\beta^k}$
5. $b_0 \leftarrow a \pmod{\beta^k}$
6. $b_1 \leftarrow \lfloor a/\beta^k \rfloor$, $b_1 \leftarrow b_1 \pmod{\beta^k}$
7. $b_2 \leftarrow \lfloor a/\beta^{2k} \rfloor$, $b_2 \leftarrow b_2 \pmod{\beta^k}$

Find the five equations for $w_0, w_1, ..., w_4$.
8. $w_0 \leftarrow a_0 \cdot b_0$
9. $w_4 \leftarrow a_2 \cdot b_2$
10. $tmp_1 \leftarrow 2 \cdot a_0$, $tmp_1 \leftarrow a_1 + tmp_1$, $tmp_1 \leftarrow 2 \cdot tmp_1$, $tmp_1 \leftarrow tmp_1 + a_2$
11. $tmp_2 \leftarrow 2 \cdot b_0$, $tmp_2 \leftarrow b_1 + tmp_2$, $tmp_2 \leftarrow 2 \cdot tmp_2$, $tmp_2 \leftarrow tmp_2 + b_2$
12. $w_1 \leftarrow tmp_1 \cdot tmp_2$
13. $tmp_1 \leftarrow 2 \cdot a_2$, $tmp_1 \leftarrow a_1 + tmp_1$, $tmp_1 \leftarrow 2 \cdot tmp_1$, $tmp_1 \leftarrow tmp_1 + a_0$
14. $tmp_2 \leftarrow 2 \cdot b_2$, $tmp_2 \leftarrow b_1 + tmp_2$, $tmp_2 \leftarrow 2 \cdot tmp_2$, $tmp_2 \leftarrow tmp_2 + b_0$
15. $w_3 \leftarrow tmp_1 \cdot tmp_2$
16. $tmp_1 \leftarrow a_0 + a_1$, $tmp_1 \leftarrow tmp_1 + a_2$, $tmp_2 \leftarrow b_0 + b_1$, $tmp_2 \leftarrow tmp_2 + b_2$
17. $w_2 \leftarrow tmp_1 \cdot tmp_2$

Continued on the next page.

Figure 5.8: Algorithm mp_toom_mul

Algorithm mp_toom_mul. This algorithm computes the product of two mp_int variables a and b using the Toom-Cook approach. Compared to the Karatsuba multiplication, this algorithm has a lower asymptotic running time of approximately $O(n^{1.464})$ but at an obvious cost in overhead. In this description, several statements have been compounded to save space. The intention is that the statements are executed from left to right across any given step.

The two inputs a and b are first split into three k-digit integers a_0, a_1, a_2 and b_0, b_1, b_2 respectively. From these smaller integers the coefficients of the polynomial basis representations $f(x)$ and $g(x)$ are known and can be used to find the relations required.

Algorithm **mp_toom_mul** (continued).
Input. mp_int a and mp_int b
Output. $c \leftarrow a \cdot b$

Now solve the system of equations.
18. $w_1 \leftarrow w_4 - w_1$, $w_3 \leftarrow w_3 - w_0$
19. $w_1 \leftarrow \lfloor w_1/2 \rfloor$, $w_3 \leftarrow \lfloor w_3/2 \rfloor$
20. $w_2 \leftarrow w_2 - w_0$, $w_2 \leftarrow w_2 - w_4$
21. $w_1 \leftarrow w_1 - w_2$, $w_3 \leftarrow w_3 - w_2$
22. $tmp_1 \leftarrow 8 \cdot w_0$, $w_1 \leftarrow w_1 - tmp_1$, $tmp_1 \leftarrow 8 \cdot w_4$, $w_3 \leftarrow w_3 - tmp_1$
23. $w_2 \leftarrow 3 \cdot w_2$, $w_2 \leftarrow w_2 - w_1$, $w_2 \leftarrow w_2 - w_3$
24. $w_1 \leftarrow w_1 - w_2$, $w_3 \leftarrow w_3 - w_2$
25. $w_1 \leftarrow \lfloor w_1/3 \rfloor, w_3 \leftarrow \lfloor w_3/3 \rfloor$

Now substitute β^k for x by shifting $w_0, w_1, ..., w_4$.
26. for n from 1 to 4 do
  26.1 $w_n \leftarrow w_n \cdot \beta^{nk}$
27. $c \leftarrow w_0 + w_1$, $c \leftarrow c + w_2$, $c \leftarrow c + w_3$, $c \leftarrow c + w_4$
28. Return(MP_OKAY)

Figure 5.9: Algorithm mp_toom_mul (continued)

The first two relations w_0 and w_4 are the points ζ_0 and ζ_∞ respectively. The relation w_1, w_2 and w_3 correspond to the points $16 \cdot \zeta_{\frac{1}{2}}, \zeta_2$ and ζ_1 respectively. These are found using logical shifts to independently find $f(y)$ and $g(y)$ which significantly speeds up the algorithm.

After the five relations $w_0, w_1, \ldots, w_4$ have been computed, the system they represent must be solved in order for the unknown coefficients w_1, w_2 and w_3 to be isolated. The steps 18 through 25 perform the system reduction required as previously described. Each step of the reduction represents the comparable matrix operation that would be performed had this been performed by pencil. For example, step 18 indicates that row 1 must be subtracted from row 4 and simultaneously row 0 subtracted from row 3.

Once the coeffients have been isolated, the polynomial $W(x) = \sum_{i=0}^{2n} w_i x^i$ is known. By substituting β^k for x, the integer result $a \cdot b$ is produced.

File: bn_mp_toom_mul.c

```
001   #ifdef BN_MP_TOOM_MUL_C
...
```

The first obvious thing to note is that this algorithm is complicated. The complexity is worth it if you are multiplying very large numbers. For example, a 10,000 digit multiplication takes approximaly 99,282,205 fewer single precision multiplications with Toom–Cook than a Comba or baseline approach (this is a savings of more than 99%). For most "crypto" sized numbers this algorithm is not practical as Karatsuba has a much lower cutoff point.

First we split a and b into three roughly equal portions. This has been accomplished (lines 38 to 69) with combinations of mp_rshd() and mp_mod_2d() function calls. At this point $a = a2 \cdot \beta^2 + a1 \cdot \beta + a0$ and similiarly for b.

Next we compute the five points $w0, w1, w2, w3$ and $w4$. Recall that $w0$ and $w4$ can be computed directly from the portions so we get those out of the way first (lines 72 and 77). Next we compute $w1, w2$ and $w3$ using Horners method.

After this point we solve for the actual values of $w1, w2$ and $w3$ by reducing the 5×5 system which is relatively straight forward.

5.2.6 Signed Multiplication

Now that algorithms to handle multiplications of every useful dimensions have been developed, a rather simple finishing touch is required. So far all of the multiplication algorithms have been unsigned multiplications which leaves only a signed multiplication algorithm to be established.

Algorithm mp_mul. This algorithm performs the signed multiplication of two inputs. It will make use of any of the three unsigned multiplication algorithms available when the input is of appropriate size. The **sign** of the result is not set until the end of the algorithm since algorithm s_mp_mul_digs will clear it.

File: bn_mp_mul.c

```
001   #ifdef BN_MP_MUL_C
...
021   #ifdef BN_MP_TOOM_MUL_C
...
026   #ifdef BN_MP_KARATSUBA_MUL_C
...
041   #ifdef BN_FAST_S_MP_MUL_DIGS_C
...
049   #ifdef BN_S_MP_MUL_DIGS_C
...
```

The implementation is rather simplistic and is not particularly noteworthy. Line 21 computes the sign of the result using the "?" operator from the C

Algorithm **mp_mul**.
Input. mp_int a and mp_int b
Output. $c \leftarrow a \cdot b$

1. If $a.sign = b.sign$ then
   1.1 $sign = MP_ZPOS$
2. else
   2.1 $sign = MP_ZNEG$
3. If $\min(a.used, b.used) \geq TOOM_MUL_CUTOFF$ then
   3.1 $c \leftarrow a \cdot b$ using algorithm mp_toom_mul
4. else if $\min(a.used, b.used) \geq KARATSUBA_MUL_CUTOFF$ then
   4.1 $c \leftarrow a \cdot b$ using algorithm mp_karatsuba_mul
5. else
   5.1 $digs \leftarrow a.used + b.used + 1$
   5.2 If $digs < MP_ARRAY$ and $\min(a.used, b.used) \leq \delta$ then
      5.2.1 $c \leftarrow a \cdot b \pmod{\beta^{digs}}$ using algorithm fast_s_mp_mul_digs.
   5.3 else
      5.3.1 $c \leftarrow a \cdot b \pmod{\beta^{digs}}$ using algorithm s_mp_mul_digs.
6. $c.sign \leftarrow sign$
7. Return the result of the unsigned multiplication performed.

Figure 5.10: Algorithm mp_mul

programming language. Line 45 computes δ using the fact that $1 << k$ is equal to 2^k.

5.3 Squaring

Squaring is a special case of multiplication where both multiplicands are equal. At first it may seem like there is no significant optimization available but in fact there is. Consider the multiplication of 576 against 241. In total there will be nine single precision multiplications performed which are $1 \cdot 6$, $1 \cdot 7$, $1 \cdot 5$, $4 \cdot 6$, $4 \cdot 7$, $4 \cdot 5$, $2 \cdot 6$, $2 \cdot 7$ and $2 \cdot 5$. Now consider the multiplication of 123 against 123. The nine products are $3 \cdot 3$, $3 \cdot 2$, $3 \cdot 1$, $2 \cdot 3$, $2 \cdot 2$, $2 \cdot 1$, $1 \cdot 3$, $1 \cdot 2$ and $1 \cdot 1$. On closer inspection some of the products are equivalent. For example, $3 \cdot 2 = 2 \cdot 3$ and $3 \cdot 1 = 1 \cdot 3$.

For any n-digit input, there are $\frac{\left(n^2+n\right)}{2}$ possible unique single precision multiplications required compared to the n^2 required for multiplication. The following

diagram gives an example of the operations required.

		1	2	3	
$\times$		1	2	3	
		$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	Row 0
	$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$		Row 1
$1 \cdot 1$	$1 \cdot 2$	$1 \cdot 3$			Row 2

Figure 5.11: Squaring Optimization Diagram

Starting from zero and numbering the columns from right to left a very simple pattern becomes obvious. For the purposes of this discussion let x represent the number being squared. The first observation is that in row k the $2k$'th column of the product has a $(x_k)^2$ term in it.

The second observation is that every column j in row k where $j \neq 2k$ is part of a double product. Every non-square term of a column will appear twice hence the name "double product". Every odd column is made up entirely of double products. In fact every column is made up of double products and at most one square (*see the exercise section*).

The third and final observation is that for row k the first unique non-square term, that is, one that hasn't already appeared in an earlier row, occurs at column $2k + 1$. For example, on row 1 of the previous squaring, column one is part of the double product with column one from row zero. Column two of row one is a square and column three is the first unique column.

5.3.1 The Baseline Squaring Algorithm

The baseline squaring algorithm is meant to be a catch-all squaring algorithm. It will handle any of the input sizes that the faster routines will not handle.

Algorithm s_mp_sqr. This algorithm computes the square of an input using the three observations on squaring. It is based fairly faithfully on algorithm 14.16 of HAC [2, pp.596-597]. Similar to algorithm s_mp_mul_digs, a temporary mp_int is allocated to hold the result of the squaring. This allows the destination mp_int to be the same as the source mp_int.

The outer loop of this algorithm begins on step 4. It is best to think of the outer loop as walking down the rows of the partial results, while the inner loop computes the columns of the partial result. Step 4.1 and 4.2 compute the square term for each row, and step 4.3 and 4.4 propagate the carry and compute the double products.

Algorithm **s_mp_sqr**.
Input. mp_int a
Output. $b \leftarrow a^2$

1. Init a temporary mp_int of at least $2 \cdot a.used + 1$ digits. (*mp_init_size*)
2. If step 1 failed return(*MP_MEM*)
3. $t.used \leftarrow 2 \cdot a.used + 1$
4. For ix from 0 to $a.used - 1$ do
   Calculate the square.
   4.1 $\hat{r} \leftarrow t_{2ix} + (a_{ix})^2$
   4.2 $t_{2ix} \leftarrow \hat{r} \pmod{\beta}$
   Calculate the double products after the square.
   4.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
   4.4 For iy from $ix + 1$ to $a.used - 1$ do
      4.4.1 $\hat{r} \leftarrow 2 \cdot a_{ix}a_{iy} + t_{ix+iy} + u$
      4.4.2 $t_{ix+iy} \leftarrow \hat{r} \pmod{\beta}$
      4.4.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
   Set the last carry.
   4.5 While $u > 0$ do
      4.5.1 $iy \leftarrow iy + 1$
      4.5.2 $\hat{r} \leftarrow t_{ix+iy} + u$
      4.5.3 $t_{ix+iy} \leftarrow \hat{r} \pmod{\beta}$
      4.5.4 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
5. Clamp excess digits of t. (*mp_clamp*)
6. Exchange b and t.
7. Clear t (*mp_clear*)
8. Return(*MP_OKAY*)

Figure 5.12: Algorithm s_mp_sqr

The requirement that a mp_word be able to represent the range $0 \le x < 2\beta^2$ arises from this very algorithm. The product $a_{ix}a_{iy}$ will lie in the range $0 \le x \le \beta^2 - 2\beta + 1$ which is obviously less than β^2 meaning that when it is multiplied by two, it can be properly represented by a mp_word.

Similar to algorithm s_mp_mul_digs, after every pass of the inner loop, the destination is correctly set to the sum of all of the partial results calculated so far. This involves expensive carry propagation which will be eliminated in the next algorithm.

File: bn_s_mp_sqr.c

```
001   #ifdef BN_S_MP_SQR_C
...
```

Inside the outer loop (line 31) the square term is calculated on line 34. The carry (line 41) has been extracted from the mp_word accumulator using a right shift. Aliases for a_{ix} and t_{ix+iy} are initialized (lines 44 and 47) to simplify the inner loop. The doubling is performed using two additions (line 56) since it is usually faster than shifting, if not at least as fast.

The important observation is that the inner loop does not begin at $iy = 0$ like for multiplication. As such the inner loops get progressively shorter as the algorithm proceeds. This is what leads to the savings compared to using a multiplication to square a number.

5.3.2 Faster Squaring by the "Comba" Method

A major drawback to the baseline method is the requirement for single precision shifting inside the $O(n^2)$ nested loop. Squaring has an additional drawback that it must double the product inside the inner loop as well. As for multiplication, the Comba technique can be used to eliminate these performance hazards.

The first obvious solution is to make an array of mp_words which will hold all of the columns. This will indeed eliminate all of the carry propagation operations from the inner loop. However, the inner product must still be doubled $O(n^2)$ times. The solution stems from the simple fact that $2a + 2b + 2c = 2(a + b + c)$. That is the sum of all of the double products is equal to double the sum of all the products. For example, $ab + ba + ac + ca = 2ab + 2ac = 2(ab + ac)$.

However, we cannot simply double all of the columns, since the squares appear only once per row. The most practical solution is to have two mp_word arrays. One array will hold the squares and the other array will hold the double products. With both arrays the doubling and carry propagation can be moved to a $O(n)$ work level outside the $O(n^2)$ level. In this case, we have an even simpler solution in mind.

Algorithm **fast_s_mp_sqr**.
Input. mp_int a
Output. $b \leftarrow a^2$

Place an array of **MP_WARRAY** mp_digits named W on the stack.
1. If $b.alloc < 2a.used + 1$ then grow b to $2a.used + 1$ digits. (*mp_grow*).
2. If step 1 failed return(*MP_MEM*).

3. $pa \leftarrow 2 \cdot a.used$
4. $\hat{W}1 \leftarrow 0$
5. for ix from 0 to $pa - 1$ do
   5.1 $_\hat{W} \leftarrow 0$
   5.2 $ty \leftarrow \text{MIN}(a.used - 1, ix)$
   5.3 $tx \leftarrow ix - ty$
   5.4 $iy \leftarrow \text{MIN}(a.used - tx, ty + 1)$
   5.5 $iy \leftarrow \text{MIN}(iy, \lfloor (ty - tx + 1)/2 \rfloor)$
   5.6 for iz from 0 to $iz - 1$ do
      5.6.1 $_\hat{W} \leftarrow _\hat{W} + a_{tx+iz}a_{ty-iz}$
   5.7 $_\hat{W} \leftarrow 2 \cdot _\hat{W} + \hat{W}1$
   5.8 if ix is even then
      5.8.1 $_\hat{W} \leftarrow _\hat{W} + \left(a_{\lfloor ix/2 \rfloor}\right)^2$
   5.9 $W_{ix} \leftarrow _\hat{W} \pmod{\beta}$
   5.10 $\hat{W}1 \leftarrow \lfloor _\hat{W}/\beta \rfloor$

6. $oldused \leftarrow b.used$
7. $b.used \leftarrow 2 \cdot a.used$
8. for ix from 0 to $pa - 1$ do
   8.1 $b_{ix} \leftarrow W_{ix}$
9. for ix from pa to $oldused - 1$ do
   9.1 $b_{ix} \leftarrow 0$
10. Clamp excess digits from b. (*mp_clamp*)
11. Return(*MP_OKAY*).

Figure 5.13: Algorithm fast_s_mp_sqr

Algorithm fast_s_mp_sqr. This algorithm computes the square of an input using the Comba technique. It is designed to be a replacement for algorithm s_mp_sqr when the number of input digits is less than **MP_WARRAY** and less than $\frac{\delta}{2}$. This algorithm is very similar to the Comba multiplier except with a few key differences we shall make note of.

First, we have an accumulator and carry variables $_\hat{W}$ and $\hat{W}1$ respectively. This is because the inner loop products are to be doubled. If we had added the previous carry in we would be doubling too much. Next we perform an addition MIN condition on iy (step 5.5) to prevent overlapping digits. For example, $a_3 \cdot a_5$ is equal $a_5 \cdot a_3$. Whereas in the multiplication case we would have $5 < a.used$ and $3 \geq 0$ is maintained since we double the sum of the products just outside the inner loop we have to avoid doing this. This is also a good thing since we perform fewer multiplications and the routine ends up being faster.

Finally the last difference is the addition of the "square" term outside the inner loop (step 5.8). We add in the square only to even outputs and it is the square of the term at the $\lfloor ix/2 \rfloor$ position.

File: bn_fast_s_mp_sqr.c

```
001   #ifdef BN_FAST_S_MP_SQR_C
...
```

This implementation is essentially a copy of Comba multiplication with the appropriate changes added to make it faster for the special case of squaring.

5.3.3 Polynomial Basis Squaring

The same algorithm that performs optimal polynomial basis multiplication can be used to perform polynomial basis squaring. The minor exception is that $\zeta_y = f(y)g(y)$ is actually equivalent to $\zeta_y = f(y)^2$ since $f(y) = g(y)$. Instead of performing $2n+1$ multiplications to find the ζ relations, squaring operations are performed instead.

5.3.4 Karatsuba Squaring

Let $f(x) = ax + b$ represent the polynomial basis representation of a number to square. Let $h(x) = (f(x))^2$ represent the square of the polynomial. The Karatsuba equation can be modified to square a number with the following equation.

$$h(x) = a^2x^2 + \left((a+b)^2 - (a^2+b^2)\right)x + b^2 \tag{5.7}$$

Upon closer inspection this equation only requires the calculation of three half-sized squares: a^2, b^2 and $(a+b)^2$. As in Karatsuba multiplication, this algorithm can be applied recursively on the input and will achieve an asymptotic running time of $O\left(n^{lg(3)}\right)$.

If the asymptotic times of Karatsuba squaring and multiplication are the same, why not simply use the multiplication algorithm instead? The answer to this arises from the cutoff point for squaring. As in multiplication there exists a cutoff point, at which the time required for a Comba based squaring and a Karatsuba based squaring meet. Due to the overhead inherent in the Karatsuba method, the cutoff point is fairly high. For example, on an AMD Athlon XP processor with $\beta = 2^{28}$, the cutoff point is around 127 digits.

Consider squaring a 200 digit number with this technique. It will be split into two 100 digit halves which are subsequently squared. The 100 digit halves will not be squared using Karatsuba, but instead using the faster Comba based squaring algorithm. If Karatsuba multiplication were used instead, the 100 digit numbers would be squared with a slower Comba based multiplication.

Algorithm **mp_karatsuba_sqr**.
Input. mp_int a
Output. $b \leftarrow a^2$

1. Initialize the following temporary mp_ints: $x0$, $x1$, $t1$, $t2$, $x0x0$ and $x1x1$.
2. If any of the initializations on step 1 failed return(MP_MEM).

Split the input. e.g. $a = x1\beta^B + x0$
3. $B \leftarrow \lfloor a.used/2 \rfloor$
4. $x0 \leftarrow a \pmod{\beta^B}$ (mp_mod_2d)
5. $x1 \leftarrow \lfloor a/\beta^B \rfloor$ (mp_lshd)

Calculate the three squares.
6. $x0x0 \leftarrow x0^2$ (mp_sqr)
7. $x1x1 \leftarrow x1^2$
8. $t1 \leftarrow x1 + x0$ (s_mp_add)
9. $t1 \leftarrow t1^2$

Compute the middle term.
10. $t2 \leftarrow x0x0 + x1x1$ (s_mp_add)
11. $t1 \leftarrow t1 - t2$

Compute final product.
12. $t1 \leftarrow t1\beta^B$ (mp_lshd)
13. $x1x1 \leftarrow x1x1\beta^{2B}$
14. $t1 \leftarrow t1 + x0x0$
15. $b \leftarrow t1 + x1x1$
16. Return(MP_OKAY).

Figure 5.14: Algorithm mp_karatsuba_sqr

Algorithm mp_karatsuba_sqr. This algorithm computes the square of an input a using the Karatsuba technique. This algorithm is very similar to the Karatsuba based multiplication algorithm with the exception that the three half-size multiplications have been replaced with three half-size squarings.

The radix point for squaring is simply placed exactly in the middle of the digits when the input has an odd number of digits, otherwise it is placed just below the middle. Step 3, 4 and 5 compute the two halves required using B as the radix point. The first two squares in steps 6 and 7 are rather straightforward while the last square is of a more compact form.

By expanding $(x1 + x0)^2$, the $x1^2$ and $x0^2$ terms in the middle disappear, that is $(x0 - x1)^2 - (x1^2 + x0^2) = 2 \cdot x0 \cdot x1$. Now if $5n$ single precision additions and a squaring of n-digits is faster than multiplying two n-digit numbers and doubling then this method is faster. Assuming no further recursions occur, the difference can be estimated with the following inequality.

Let p represent the cost of a single precision addition and q the cost of a single precision multiplication both in terms of time[3].

$$5pn + \frac{q(n^2 + n)}{2} \leq pn + qn^2 \tag{5.8}$$

For example, on an AMD Athlon XP processor $p = \frac{1}{3}$ and $q = 6$. This implies that the following inequality should hold.

$$\begin{aligned} \tfrac{5n}{3} + 3n^2 + 3n &< \tfrac{n}{3} + 6n^2 \\ \tfrac{5}{3} + 3n + 3 &< \tfrac{1}{3} + 6n \\ \tfrac{13}{9} &< n \end{aligned}$$

This results in a cutoff point around $n = 2$. As a consequence it is actually faster to compute the middle term the "long way" on processors where multiplication is substantially slower[4] than simpler operations such as addition.

File: bn_mp_karatsuba_sqr.c

```
001   #ifdef BN_MP_KARATSUBA_SQR_C
...
```

This implementation is largely based on the implementation of algorithm mp_karatsuba_mul. It uses the same inline style to copy and shift the input into the two halves. The loop from line 51 to line 67 has been modified since only one input exists. The **used** count of both $x0$ and $x1$ is fixed up and $x0$ is clamped before the calculations begin. At this point $x1$ and $x0$ are valid equivalents to the respective halves as if mp_rshd and mp_mod_2d had been used.

By inlining the copy and shift operations the cutoff point for Karatsuba multiplication can be lowered. On the Athlon the cutoff point is exactly at the point where Comba squaring can no longer be used (*128 digits*). On slower processors such as the Intel P4 it is actually below the Comba limit (*at 110 digits*).

[3] Or machine clock cycles.

[4] On the Athlon there is a 1:17 ratio between clock cycles for addition and multiplication. On the Intel P4 processor this ratio is 1:29 making this method even more beneficial. The only common exception is the ARMv4 processor which has a ratio of 1:7.

This routine uses the same error trap coding style as mp_karatsuba_sqr. As the temporary variables are initialized errors are redirected to the error trap higher up. If the algorithm completes without error the error code is set to **MP_OKAY** and mp_clears are executed normally.

5.3.5 Toom-Cook Squaring

The Toom-Cook squaring algorithm mp_toom_sqr is heavily based on the algorithm mp_toom_mul with the exception that squarings are used instead of multiplication to find the five relations. The reader is encouraged to read the description of the latter algorithm and try to derive their own Toom-Cook squaring algorithm.

5.3.6 High Level Squaring

Algorithm **mp_sqr**.
Input. mp_int a
Output. $b \leftarrow a^2$

1. If $a.used \geq TOOM_SQR_CUTOFF$ then
   1.1 $b \leftarrow a^2$ using algorithm mp_toom_sqr
2. else if $a.used \geq KARATSUBA_SQR_CUTOFF$ then
   2.1 $b \leftarrow a^2$ using algorithm mp_karatsuba_sqr
3. else
   3.1 $digs \leftarrow a.used + b.used + 1$
   3.2 If $digs < MP_ARRAY$ and $a.used \leq \delta$ then
      3.2.1 $b \leftarrow a^2$ using algorithm fast_s_mp_sqr.
   3.3 else
      3.3.1 $b \leftarrow a^2$ using algorithm s_mp_sqr.
4. $b.sign \leftarrow MP_ZPOS$
5. Return the result of the unsigned squaring performed.

Figure 5.15: Algorithm mp_sqr

Algorithm mp_sqr. This algorithm computes the square of the input using one of four different algorithms. If the input is very large and has at least **TOOM_SQR_CUTOFF** or **KARATSUBA_SQR_CUTOFF** digits then either the Toom-Cook or the Karatsuba Squaring algorithm is used. If neither of the polynomial basis algorithms should be used then either the Comba or baseline algorithm is used.

File: bn_mp_sqr.c

```
001   #ifdef BN_MP_SQR_C
...
019   #ifdef BN_MP_TOOM_SQR_C
...
026   #ifdef BN_MP_KARATSUBA_SQR_C
...
032   #ifdef BN_FAST_S_MP_SQR_C
...
041   #ifdef BN_S_MP_SQR_C
...
```

Exercises

[3] Devise an efficient algorithm for selection of the radix point to handle inputs that have different number of digits in Karatsuba multiplication.

[2] In section 5.3 the fact that every column of a squaring is made up of double products and at most one square is stated. Prove this statement.

[3] Prove the equation for Karatsuba squaring.

[1] Prove that Karatsuba squaring requires $O\left(n^{lg(3)}\right)$ time.

[2] Determine the minimal ratio between addition and multiplication clock cycles required for equation 6.7 to be true.

[3] Implement a threaded version of Comba multiplication (and squaring) where you compute subsets of the columns in each thread. Determine a cutoff point where it is effective and add the logic to mp_mul() and mp_sqr().

[4] Same as the previous but also modify the Karatsuba and Toom-Cook. You must increase the throughput of mp_exptmod() for random odd moduli in the range $512 \ldots 4096$ bits significantly ($> 2x$) to complete this challenge.

Chapter 6

Modular Reduction

6.1 Basics of Modular Reduction

Modular reduction is an operation that arises quite often within public key cryptography algorithms and various number theoretic algorithms, such as factoring. Modular reduction algorithms are the third class of algorithms of the "multipliers" set. A number a is said to be *reduced* modulo another number b by finding the remainder of the division a/b. Full integer division with remainder is a topic to be covered in 8.1.

Modular reduction is equivalent to solving for r in the following equation. $a = bq + r$ where $q = \lfloor a/b \rfloor$. The result r is said to be "congruent to a modulo b" which is also written as $r \equiv a \pmod{b}$. In other vernacular r is known as the "modular residue" which leads to "quadratic residue"[1] and other forms of residues.

Modular reductions are normally used to create either finite groups, rings or fields. The most common usage for performance driven modular reductions is in modular exponentiation algorithms. That is to compute $d = a^b \pmod{c}$ as fast as possible. This operation is used in the RSA and Diffie-Hellman public key algorithms, for example. Modular multiplication and squaring also appears as a fundamental operation in elliptic curve cryptographic algorithms. As will be discussed in the subsequent chapter there exist fast algorithms for computing modular exponentiations without having to perform (*in this example*) $b - 1$ multiplications. These algorithms will produce partial results in the

[1] That's fancy talk for $b \equiv a^2 \pmod{p}$.

range $0 \leq x < c^2$ which can be taken advantage of to create several efficient algorithms. They have also been used to create redundancy check algorithms known as CRCs, error correction codes such as Reed-Solomon and solve a variety of number theoeretic problems.

6.2 The Barrett Reduction

The Barrett reduction algorithm [7] was inspired by fast division algorithms which multiply by the reciprocal to emulate division. Barretts observation was that the residue c of a modulo b is equal to

$$c = a - b \cdot \lfloor a/b \rfloor \tag{6.1}$$

Since algorithms such as modular exponentiation would be using the same modulus extensively, typical DSP[2] intuition would indicate the next step would be to replace a/b by a multiplication by the reciprocal. However, DSP intuition on its own will not work as these numbers are considerably larger than the precision of common DSP floating point data types. It would take another common optimization to optimize the algorithm.

6.2.1 Fixed Point Arithmetic

The trick used to optimize the above equation is based on a technique of emulating floating point data types with fixed precision integers. Fixed point arithmetic would become very popular as it greatly optimize the "3d-shooter" genre of games in the mid 1990s when floating point units were fairly slow if not unavailable. The idea behind fixed point arithmetic is to take a normal k-bit integer data type and break it into p-bit integer and a q-bit fraction part (*where* $p + q = k$).

In this system a k-bit integer n would actually represent $n/2^q$. For example, with $q = 4$ the integer $n = 37$ would actually represent the value 2.3125. To multiply two fixed point numbers the integers are multiplied using traditional arithmetic and subsequently normalized by moving the implied decimal point back to where it should be. For example, with $q = 4$ to multiply the integers 9 and 5 they must be converted to fixed point first by multiplying by 2^q. Let $a = 9(2^q)$ represent the fixed point representation of 9 and $b = 5(2^q)$ represent

[2]It is worth noting that Barrett's paper targeted the DSP56K processor.

the fixed point representation of 5. The product ab is equal to $45(2^{2q})$ which when normalized by dividing by 2^q produces $45(2^q)$.

This technique became popular since a normal integer multiplication and logical shift right are the only required operations to perform a multiplication of two fixed point numbers. Using fixed point arithmetic, division can be easily approximated by multiplying by the reciprocal. If 2^q is equivalent to one than $2^q/b$ is equivalent to the fixed point approximation of $1/b$ using real arithmetic. Using this fact dividing an integer a by another integer b can be achieved with the following expression.

$$\lfloor a/b \rfloor \sim \lfloor (a \cdot \lfloor 2^q/b \rfloor)/2^q \rfloor \tag{6.2}$$

The precision of the division is proportional to the value of q. If the divisor b is used frequently as is the case with modular exponentiation pre-computing $2^q/b$ will allow a division to be performed with a multiplication and a right shift. Both operations are considerably faster than division on most processors.

Consider dividing 19 by 5. The correct result is $\lfloor 19/5 \rfloor = 3$. With $q = 3$ the reciprocal is $\lfloor 2^q/5 \rfloor = 1$ which leads to a product of 19 which when divided by 2^q produces 2. However, with $q = 4$ the reciprocal is $\lfloor 2^q/5 \rfloor = 3$ and the result of the emulated division is $\lfloor 3 \cdot 19/2^q \rfloor = 3$ which is correct. The value of 2^q must be close to or ideally larger than the dividend. In effect if a is the dividend then q should allow $0 \le \lfloor a/2^q \rfloor \le 1$ in order for this approach to work correctly. Plugging this form of divison into the original equation the following modular residue equation arises.

$$c = a - b \cdot \lfloor (a \cdot \lfloor 2^q/b \rfloor)/2^q \rfloor \tag{6.3}$$

Using the notation from [7] the value of $\lfloor 2^q/b \rfloor$ will be represented by the μ symbol. Using the μ variable also helps re-inforce the idea that it is meant to be computed once and re-used.

$$c = a - b \cdot \lfloor (a \cdot \mu)/2^q \rfloor \tag{6.4}$$

Provided that $2^q \ge a$ this algorithm will produce a quotient that is either exactly correct or off by a value of one. In the context of Barrett reduction the value of a is bound by $0 \le a \le (b-1)^2$ meaning that $2^q \ge b^2$ is sufficient to ensure the reciprocal will have enough precision.

Let n represent the number of digits in b. This algorithm requires approximately $2n^2$ single precision multiplications to produce the quotient and another n^2 single precision multiplications to find the residue. In total $3n^2$ single precision multiplications are required to reduce the number.

For example, if $b = 1179677$ and $q = 41$ ($2^q > b^2$), then the reciprocal μ is equal to $\lfloor 2^q/b \rfloor = 1864089$. Consider reducing $a = 180388626447$ modulo b using the above reduction equation. The quotient using the new formula is $\lfloor (a \cdot \mu)/2^q \rfloor = 152913$. By subtracting $152913b$ from a the correct residue $a \equiv 677346 \pmod{b}$ is found.

6.2.2 Choosing a Radix Point

Using the fixed point representation a modular reduction can be performed with $3n^2$ single precision multiplications. If that were the best that could be achieved a full division[3] might as well be used in its place. The key to optimizing the reduction is to reduce the precision of the initial multiplication that finds the quotient.

Let a represent the number of which the residue is sought. Let b represent the modulus used to find the residue. Let m represent the number of digits in b. For the purposes of this discussion we will assume that the number of digits in a is $2m$, which is generally true if two m-digit numbers have been multiplied. Dividing a by b is the same as dividing a $2m$ digit integer by a m digit integer. Digits below the $m-1$'th digit of a will contribute at most a value of 1 to the quotient because $\beta^k < b$ for any $0 \le k \le m-1$. Another way to express this is by re-writing a as two parts. If $a' \equiv a \pmod{b^m}$ and $a'' = a - a'$ then $\frac{a}{b} \equiv \frac{a'+a''}{b}$ which is equivalent to $\frac{a'}{b} + \frac{a''}{b}$. Since a' is bound to be less than b the quotient is bound by $0 \le \frac{a'}{b} < 1$.

Since the digits of a' do not contribute much to the quotient the observation is that they might as well be zero. However, if the digits "might as well be zero" they might as well not be there in the first place. Let $q_0 = \lfloor a/\beta^{m-1} \rfloor$ represent the input with the irrelevant digits trimmed. Now the modular reduction is trimmed to the almost equivalent equation

$$c = a - b \cdot \lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor \tag{6.5}$$

Note that the original divisor 2^q has been replaced with β^{m+1} where in this case q is a multiple of $lg(\beta)$. Also note that the exponent on the divisor when added to the amount q_0 was shifted by equals $2m$. If the optimization had not been performed the divisor would have the exponent $2m$ so in the end the exponents do "add up". Using the above equation the quotient $\lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor$ can be off from the true quotient by at most two. The original fixed point

[3] A division requires approximately $O(2cn^2)$ single precision multiplications for a small value of c. See 8.1 for further details.

quotient can be off by as much as one (*provided the radix point is chosen suitably*) and now that the lower irrelevent digits have been trimmed the quotient can be off by an additional value of one for a total of at most two. This implies that $0 \le a - b \cdot \lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor < 3b$. By first subtracting b times the quotient and then conditionally subtracting b once or twice the residue is found.

The quotient is now found using $(m+1)(m) = m^2 + m$ single precision multiplications and the residue with an additional m^2 single precision multiplications, ignoring the subtractions required. In total $2m^2 + m$ single precision multiplications are required to find the residue. This is considerably faster than the original attempt.

For example, let $\beta = 10$ represent the radix of the digits. Let $b = 9999$ represent the modulus which implies $m = 4$. Let $a = 99929878$ represent the value of which the residue is desired. In this case $q = 8$ since $10^7 < 9999^2$ meaning that $\mu = \lfloor \beta^q / b \rfloor = 10001$. With the new observation the multiplicand for the quotient is equal to $q_0 = \lfloor a/\beta^{m-1} \rfloor = 99929$. The quotient is then $\lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor = 9993$. Subtracting $9993b$ from a and the correct residue $a \equiv 9871 \pmod{b}$ is found.

6.2.3 Trimming the Quotient

So far the reduction algorithm has been optimized from $3m^2$ single precision multiplications down to $2m^2 + m$ single precision multiplications. As it stands now the algorithm is already fairly fast compared to a full integer division algorithm. However, there is still room for optimization.

After the first multiplication inside the quotient $(q_0 \cdot \mu)$ the value is shifted right by $m+1$ places effectively nullifying the lower half of the product. It would be nice to be able to remove those digits from the product to effectively cut down the number of single precision multiplications. If the number of digits in the modulus m is far less than β a full product is not required for the algorithm to work properly. In fact the lower $m-2$ digits will not affect the upper half of the product at all and do not need to be computed.

The value of μ is a m-digit number and q_0 is a $m+1$ digit number. Using a full multiplier $(m+1)(m) = m^2 + m$ single precision multiplications would be required. Using a multiplier that will only produce digits at and above the $m-1$'th digit reduces the number of single precision multiplications to $\frac{m^2+m}{2}$ single precision multiplications.

6.2.4 Trimming the Residue

After the quotient has been calculated it is used to reduce the input. As previously noted the algorithm is not exact and it can be off by a small multiple of the modulus, that is $0 \le a - b \cdot \lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor < 3b$. If b is m digits than the result of reduction equation is a value of at most $m + 1$ digits (*provided* $3 < \beta$) implying that the upper $m - 1$ digits are implicitly zero.

The next optimization arises from this very fact. Instead of computing $b \cdot \lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor$ using a full $O(m^2)$ multiplication algorithm only the lower $m + 1$ digits of the product have to be computed. Similarly the value of a can be reduced modulo β^{m+1} before the multiple of b is subtracted which simplifes the subtraction as well. A multiplication that produces only the lower $m + 1$ digits requires $\frac{m^2+3m-2}{2}$ single precision multiplications.

With both optimizations in place the algorithm is the algorithm Barrett proposed. It requires $m^2 + 2m - 1$ single precision multiplications which is considerably faster than the straightforward $3m^2$ method.

6.2.5 The Barrett Algorithm

Algorithm **mp_reduce**.
Input. mp_int a, mp_int b and $\mu = \lfloor \beta^{2m}/b \rfloor, m = \lceil lg_\beta(b) \rceil, (0 \le a < b^2, b > 1)$
Output. $a \pmod b$

Let m represent the number of digits in b.
1. Make a copy of a and store it in q. (*mp_init_copy*)
2. $q \leftarrow \lfloor q/\beta^{m-1} \rfloor$ (*mp_rshd*)

Produce the quotient.
3. $q \leftarrow q \cdot \mu$ (*note: only produce digits at or above* $m-1$)
4. $q \leftarrow \lfloor q/\beta^{m+1} \rfloor$

Subtract the multiple of modulus from the input.
5. $a \leftarrow a \pmod{\beta^{m+1}}$ (*mp_mod_2d*)
6. $q \leftarrow q \cdot b \pmod{\beta^{m+1}}$ (*s_mp_mul_digs*)
7. $a \leftarrow a - q$ (*mp_sub*)

Add β^{m+1} if a carry occured.
8. If $a < 0$ then (*mp_cmp_d*)
   8.1 $q \leftarrow 1$ (*mp_set*)
   8.2 $q \leftarrow q \cdot \beta^{m+1}$ (*mp_lshd*)
   8.3 $a \leftarrow a + q$

Now subtract the modulus if the residue is too large (e.g. quotient too small).
9. While $a \ge b$ do (*mp_cmp*)
   9.1 $c \leftarrow a - b$
10. Clear q.
11. Return(*MP_OKAY*)

Figure 6.1: Algorithm mp_reduce

Algorithm mp_reduce. This algorithm will reduce the input a modulo b in place using the Barrett algorithm. It is loosely based on algorithm 14.42 of HAC [2, pp. 602] which is based on the paper from Paul Barrett [7]. The algorithm has several restrictions and assumptions which must be adhered to for the algorithm to work.

First the modulus b is assumed to be positive and greater than one. If the modulus were less than or equal to one than subtracting a multiple of it would either accomplish nothing or actually enlarge the input. The input a must be in the range $0 \le a < b^2$ in order for the quotient to have enough precision. If a

is the product of two numbers that were already reduced modulo b, this will not be a problem. Technically the algorithm will still work if $a \geq b^2$ but it will take much longer to finish. The value of μ is passed as an argument to this algorithm and is assumed to be calculated and stored before the algorithm is used.

Recall that the multiplication for the quotient on step 3 must only produce digits at or above the $m-1$'th position. An algorithm called *s_mp_mul_high_digs* which has not been presented is used to accomplish this task. The algorithm is based on *s_mp_mul_digs* except that instead of stopping at a given level of precision it starts at a given level of precision. This optimal algorithm can only be used if the number of digits in b is very much smaller than β.

While it is known that $a \geq b \cdot \lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor$ only the lower $m+1$ digits are being used to compute the residue, so an implied "borrow" from the higher digits might leave a negative result. After the multiple of the modulus has been subtracted from a the residue must be fixed up in case it is negative. The invariant β^{m+1} must be added to the residue to make it positive again.

The while loop at step 9 will subtract b until the residue is less than b. If the algorithm is performed correctly this step is performed at most twice, and on average once. However, if $a \geq b^2$ than it will iterate substantially more times than it should.

File: bn_mp_reduce.c

```
001   #ifdef BN_MP_REDUCE_C
...
037   #ifdef BN_S_MP_MUL_HIGH_DIGS_C
...
```

The first multiplication that determines the quotient can be performed by only producing the digits from $m-1$ and up. This essentially halves the number of single precision multiplications required. However, the optimization is only safe if β is much larger than the number of digits in the modulus. In the source code this is evaluated on lines 38 to 45 where algorithm s_mp_mul_high_digs is used when it is safe to do so.

6.2.6 The Barrett Setup Algorithm

In order to use algorithm mp_reduce the value of μ must be calculated in advance. Ideally this value should be computed once and stored for future use so that the Barrett algorithm can be used without delay.

Algorithm **mp_reduce_setup**.
Input. mp_int a $(a > 1)$
Output. $\mu \leftarrow \lfloor \beta^{2m}/a \rfloor$

1. $\mu \leftarrow 2^{2 \cdot lg(\beta) \cdot m}$ (mp_2expt)
2. $\mu \leftarrow \lfloor \mu/b \rfloor$ (mp_div)
3. Return(MP_OKAY)

Figure 6.2: Algorithm mp_reduce_setup

Algorithm mp_reduce_setup. This algorithm computes the reciprocal μ required for Barrett reduction. First β^{2m} is calculated as $2^{2 \cdot lg(\beta) \cdot m}$ which is equivalent and much faster. The final value is computed by taking the integer quotient of $\lfloor \mu/b \rfloor$.

File: bn_mp_reduce_setup.c

```
001   #ifdef BN_MP_REDUCE_SETUP_C
...
```

This simple routine calculates the reciprocal μ required by Barrett reduction. Note the extended usage of algorithm mp_div where the variable which would received the remainder is passed as NULL. As will be discussed in 8.1 the division routine allows both the quotient and the remainder to be passed as NULL meaning to ignore the value.

6.3 The Montgomery Reduction

Montgomery reduction[4] [8] is by far the most interesting form of reduction in common use. It computes a modular residue which is not actually equal to the residue of the input yet instead equal to a residue times a constant. However, as perplexing as this may sound the algorithm is relatively simple and very efficient.

Throughout this entire section the variable n will represent the modulus used to form the residue. As will be discussed shortly the value of n must be odd. The variable x will represent the quantity of which the residue is sought. Similar to the Barrett algorithm the input is restricted to $0 \le x < n^2$. To begin the description some simple number theory facts must be established.

[4]Thanks to Niels Ferguson for his insightful explanation of the algorithm.

Fact 1. Adding n to x does not change the residue since in effect it adds one to the quotient $\lfloor x/n \rfloor$. Another way to explain this is that n is (*or multiples of* n *are*) congruent to zero modulo n. Adding zero will not change the value of the residue.

Fact 2. If x is even then performing a division by two in $\mathbb{Z}$ is congruent to $x \cdot 2^{-1} \pmod{n}$. Actually this is an application of the fact that if x is evenly divisible by any $k \in \mathbb{Z}$ then division in $\mathbb{Z}$ will be congruent to multiplication by k^{-1} modulo n.

From these two simple facts the following simple algorithm can be derived.

Algorithm **Montgomery Reduction**.
Input. Integer x, n and k
Output. $2^{-k}x \pmod{n}$

1. for t from 1 to k do
   1.1 If x is odd then
      1.1.1 $x \leftarrow x + n$
   1.2 $x \leftarrow x/2$
2. Return x.

Figure 6.3: Algorithm Montgomery Reduction

The algorithm reduces the input one bit at a time using the two congruencies stated previously. Inside the loop n, which is odd, is added to x if x is odd. This forces x to be even which allows the division by two in $\mathbb{Z}$ to be congruent to a modular division by two. Since x is assumed to be initially much larger than n the addition of n will contribute an insignificant magnitude to x. Let r represent the final result of the Montgomery algorithm. If $k > lg(n)$ and $0 \leq x < n^2$ then the final result is limited to $0 \leq r < \lfloor x/2^k \rfloor + n$. As a result at most a single subtraction is required to get the residue desired.

Step number (t)	**Result** (x)
1	$x + n = 5812$, $x/2 = 2906$
2	$x/2 = 1453$
3	$x + n = 1710$, $x/2 = 855$
4	$x + n = 1112$, $x/2 = 556$
5	$x/2 = 278$
6	$x/2 = 139$
7	$x + n = 396$, $x/2 = 198$
8	$x/2 = 99$
9	$x + n = 356$, $x/2 = 178$

Figure 6.4: Example of Montgomery Reduction (I)

Consider the example in figure 6.4 which reduces $x = 5555$ modulo $n = 257$ when $k = 9$ (note $\beta^k = 512$ which is larger than n). The result of the algorithm $r = 178$ is congruent to the value of $2^{-9} \cdot 5555 \pmod{257}$. When r is multiplied by 2^9 modulo 257 the correct residue $r \equiv 158$ is produced.

Let $k = \lfloor lg(n) \rfloor + 1$ represent the number of bits in n. The current algorithm

requires $2k^2$ single precision shifts and k^2 single precision additions. At this rate the algorithm is most certainly slower than Barrett reduction and not terribly useful. Fortunately there exists an alternative representation of the algorithm.

Algorithm **Montgomery Reduction** (modified I).
Input. Integer x, n and k ($2^k > n$)
Output. $2^{-k}x \pmod n$

1. for t from 1 to k do
   1.1 If the t'th bit of x is one then
      1.1.1 $x \leftarrow x + 2^t n$
2. Return $x/2^k$.

Figure 6.5: Algorithm Montgomery Reduction (modified I)

This algorithm is equivalent since $2^t n$ is a multiple of n and the lower k bits of x are zero by step 2. The number of single precision shifts has now been reduced from $2k^2$ to $k^2 + k$ which is only a small improvement.

Step number (t)	**Result** (x)	**Result** (x) **in Binary**
–	5555	1010110110011
1	$x + 2^0 n = 5812$	1011010110100
2	5812	1011010110100
3	$x + 2^2 n = 6840$	1101010111000
4	$x + 2^3 n = 8896$	10001011000000
5	8896	10001011000000
6	8896	10001011000000
7	$x + 2^6 n = 25344$	110001100000000
8	25344	110001100000000
9	$x + 2^7 n = 91136$	10110010000000000
–	$x/2^k = 178$	

Figure 6.6: Example of Montgomery Reduction (II)

Figure 6.6 demonstrates the modified algorithm reducing $x = 5555$ modulo $n = 257$ with $k = 9$. With this algorithm a single shift right at the end is the only right shift required to reduce the input instead of k right shifts inside the loop. Note that for the iterations $t = 2, 5, 6$ and 8 where the result x is not changed. In those iterations the t'th bit of x is zero and the appropriate

multiple of n does not need to be added to force the t'th bit of the result to zero.

6.3.1 Digit Based Montgomery Reduction

Instead of computing the reduction on a bit-by-bit basis it is actually much faster to compute it on digit-by-digit basis. Consider the previous algorithm re-written to compute the Montgomery reduction in this new fashion.

Algorithm **Montgomery Reduction** (modified II).
Input. Integer x, n and k ($\beta^k > n$)
Output. $\beta^{-k}x \pmod n$

1. for t from 0 to $k-1$ do
   1.1 $x \leftarrow x + \mu n \beta^t$
2. Return x/β^k.

Figure 6.7: Algorithm Montgomery Reduction (modified II)

The value $\mu n \beta^t$ is a multiple of the modulus n meaning that it will not change the residue. If the first digit of the value $\mu n \beta^t$ equals the negative (modulo β) of the t'th digit of x then the addition will result in a zero digit. This problem breaks down to solving the following congruency.

$$\begin{aligned} x_t + \mu n_0 &\equiv 0 \pmod \beta \\ \mu n_0 &\equiv -x_t \pmod \beta \\ \mu &\equiv -x_t/n_0 \pmod \beta \end{aligned}$$

In each iteration of the loop on step 1 a new value of μ must be calculated. The value of $-1/n_0 \pmod \beta$ is used extensively in this algorithm and should be precomputed. Let ρ represent the negative of the modular inverse of n_0 modulo β.

For example, let $\beta = 10$ represent the radix. Let $n = 17$ represent the modulus which implies $k = 2$ and $\rho \equiv 7$. Let $x = 33$ represent the value to reduce.

Step (t)	**Value of** x	**Value of** μ
–	33	–
0	$33 + \mu n = 50$	1
1	$50 + \mu n\beta = 900$	5

Figure 6.8: Example of Montgomery Reduction

The final result 900 is then divided by β^k to produce the final result 9. The first observation is that $9 \not\equiv x \pmod{n}$ which implies the result is not the modular residue of x modulo n. However, recall that the residue is actually multiplied by β^{-k} in the algorithm. To get the true residue the value must be multiplied by β^k. In this case $\beta^k \equiv 15 \pmod{n}$ and the correct residue is $9 \cdot 15 \equiv 16 \pmod{n}$.

6.3.2 Baseline Montgomery Reduction

The baseline Montgomery reduction algorithm will produce the residue for any size input. It is designed to be a catch-all algororithm for Montgomery reductions.

Algorithm **mp_montgomery_reduce**.
Input. mp_int x, mp_int n and a digit $\rho \equiv -1/n_0 \pmod{n}$.
$(0 \le x < n^2, n > 1, (n, \beta) = 1, \beta^k > n)$
Output. $\beta^{-k}x \pmod{n}$

1. $digs \leftarrow 2n.used + 1$
2. If $digs < MP_ARRAY$ and $m.used < \delta$ then
   2.1 Use algorithm fast_mp_montgomery_reduce instead.

Setup x for the reduction.
3. If $x.alloc < digs$ then grow x to $digs$ digits.
4. $x.used \leftarrow digs$

Eliminate the lower k digits.
5. For ix from 0 to $k - 1$ do
   5.1 $\mu \leftarrow x_{ix} \cdot \rho \pmod{\beta}$
   5.2 $u \leftarrow 0$
   5.3 For iy from 0 to $k - 1$ do
      5.3.1 $\hat{r} \leftarrow \mu n_{iy} + x_{ix+iy} + u$
      5.3.2 $x_{ix+iy} \leftarrow \hat{r} \pmod{\beta}$
      5.3.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
   5.4 While $u > 0$ do
      5.4.1 $iy \leftarrow iy + 1$
      5.4.2 $x_{ix+iy} \leftarrow x_{ix+iy} + u$
      5.4.3 $u \leftarrow \lfloor x_{ix+iy}/\beta \rfloor$
      5.4.4 $x_{ix+iy} \leftarrow x_{ix+iy} \pmod{\beta}$

Divide by β^k and fix up as required.
6. $x \leftarrow \lfloor x/\beta^k \rfloor$
7. If $x \ge n$ then
   7.1 $x \leftarrow x - n$
8. Return(MP_OKAY).

Figure 6.9: Algorithm mp_montgomery_reduce

Algorithm mp_montgomery_reduce. This algorithm reduces the input x modulo n in place using the Montgomery reduction algorithm. The algorithm is loosely based on algorithm 14.32 of [2, pp.601] except it merges the multiplication of $\mu n \beta^t$ with the addition in the inner loop. The restrictions on this algorithm are fairly easy to adapt to. First $0 \le x < n^2$ bounds the input to

numbers in the same range as for the Barrett algorithm. Additionally if $n > 1$ and n is odd there will exist a modular inverse ρ. ρ must be calculated in advance of this algorithm. Finally the variable k is fixed and a pseudonym for $n.used$.

Step 2 decides whether a faster Montgomery algorithm can be used. It is based on the Comba technique meaning that there are limits on the size of the input. This algorithm is discussed in sub-section 6.3.3.

Step 5 is the main reduction loop of the algorithm. The value of μ is calculated once per iteration in the outer loop. The inner loop calculates $x + \mu n\beta^{ix}$ by multiplying μn and adding the result to x shifted by ix digits. Both the addition and multiplication are performed in the same loop to save time and memory. Step 5.4 will handle any additional carries that escape the inner loop.

Using a quick inspection this algorithm requires n single precision multiplications for the outer loop and n^2 single precision multiplications in the inner loop. In total $n^2 + n$ single precision multiplications which compares favourably to Barrett at $n^2 + 2n - 1$ single precision multiplications.

File: bn_mp_montgomery_reduce.c

```
001   #ifdef BN_MP_MONTGOMERY_REDUCE_C
...
```

This is the baseline implementation of the Montgomery reduction algorithm. Lines 28 to 33 determine if the Comba based routine can be used instead. Line 46 computes the value of μ for that particular iteration of the outer loop.

The multiplication $\mu n\beta^{ix}$ is performed in one step in the inner loop. The alias $tmpx$ refers to the ix'th digit of x and the alias $tmpn$ refers to the modulus n.

6.3.3 Faster "Comba" Montgomery Reduction

The Montgomery reduction requires fewer single precision multiplications than a Barrett reduction, however it is much slower due to the serial nature of the inner loop. The Barrett reduction algorithm requires two slightly modified multipliers which can be implemented with the Comba technique. The Montgomery reduction algorithm cannot directly use the Comba technique to any significant advantage since the inner loop calculates a $k \times 1$ product k times.

The biggest obstacle is that at the ix'th iteration of the outer loop the value of x_{ix} is required to calculate μ. This means the carries from 0 to $ix - 1$ must have been propagated upwards to form a valid ix'th digit. The solution as it

turns out is very simple. Perform a Comba like multiplier and inside the outer loop just after the inner loop fix up the $ix + 1$'th digit by forwarding the carry.

With this change in place the Montgomery reduction algorithm can be performed with a Comba style multiplication loop which substantially increases the speed of the algorithm.

Algorithm **fast_mp_montgomery_reduce**.
Input. mp_int x, mp_int n and a digit $\rho \equiv -1/n_0 \pmod n$.
$(0 \le x < n^2, n > 1, (n, \beta) = 1, \beta^k > n)$
Output. $\beta^{-k}x \pmod n$

Place an array of **MP_WARRAY** mp_word variables called $\hat{W}$ on the stack.
1. if $x.alloc < n.used + 1$ then grow x to $n.used + 1$ digits.
Copy the digits of x into the array $\hat{W}$
2. For ix from 0 to $x.used - 1$ do
  2.1 $\hat{W}_{ix} \leftarrow x_{ix}$
3. For ix from $x.used$ to $2n.used - 1$ do
  3.1 $\hat{W}_{ix} \leftarrow 0$
Elimiate the lower k digits.
4. for ix from 0 to $n.used - 1$ do
  4.1 $\mu \leftarrow \hat{W}_{ix} \cdot \rho \pmod \beta$
  4.2 For iy from 0 to $n.used - 1$ do
    4.2.1 $\hat{W}_{iy+ix} \leftarrow \hat{W}_{iy+ix} + \mu \cdot n_{iy}$
  4.3 $\hat{W}_{ix+1} \leftarrow \hat{W}_{ix+1} + \lfloor \hat{W}_{ix}/\beta \rfloor$
Propagate carries upwards.
5. for ix from $n.used$ to $2n.used + 1$ do
  5.1 $\hat{W}_{ix+1} \leftarrow \hat{W}_{ix+1} + \lfloor \hat{W}_{ix}/\beta \rfloor$
Shift right and reduce modulo β simultaneously.
6. for ix from 0 to $n.used + 1$ do
  6.1 $x_{ix} \leftarrow \hat{W}_{ix+n.used} \pmod \beta$
Zero excess digits and fixup x.
7. if $x.used > n.used + 1$ then do
  7.1 for ix from $n.used + 1$ to $x.used - 1$ do
    7.1.1 $x_{ix} \leftarrow 0$
8. $x.used \leftarrow n.used + 1$
9. Clamp excessive digits of x.
10. If $x \ge n$ then
  10.1 $x \leftarrow x - n$
11. Return(*MP_OKAY*).

Figure 6.10: Algorithm fast_mp_montgomery_reduce

Algorithm fast_mp_montgomery_reduce. This algorithm will compute the Montgomery reduction of x modulo n using the Comba technique. It is on most computer platforms significantly faster than algorithm mp_montgomery_reduce and algorithm mp_reduce (*Barrett reduction*). The algorithm has the same re-

strictions on the input as the baseline reduction algorithm. An additional two restrictions are imposed on this algorithm. The number of digits k in the the modulus n must not violate $MP_WARRAY > 2k+1$ and $n < \delta$. When $\beta = 2^{28}$ this algorithm can be used to reduce modulo a modulus of at most $3,556$ bits in length.

As in the other Comba reduction algorithms there is a $\hat{W}$ array which stores the columns of the product. It is initially filled with the contents of x with the excess digits zeroed. The reduction loop is very similar the to the baseline loop at heart. The multiplication on step 4.1 can be single precision only since $ab \pmod{\beta} \equiv (a \bmod \beta)(b \bmod \beta)$. Some multipliers such as those on the ARM processors take a variable length time to complete depending on the number of bytes of result it must produce. By performing a single precision multiplication instead half the amount of time is spent.

Also note that digit $\hat{W}_{ix}$ must have the carry from the $ix-1$'th digit propagated upwards in order for this to work. That is what step 4.3 will do. In effect over the $n.used$ iterations of the outer loop the $n.used$'th lower columns all have the their carries propagated forwards. Note how the upper bits of those same words are not reduced modulo β. This is because those values will be discarded shortly and there is no point.

Step 5 will propagate the remainder of the carries upwards. On step 6 the columns are reduced modulo β and shifted simultaneously as they are stored in the destination x.

File: bn_fast_mp_montgomery_reduce.c

```
001   #ifdef BN_FAST_MP_MONTGOMERY_REDUCE_C
...
```

The $\hat{W}$ array is first filled with digits of x on line 49 then the rest of the digits are zeroed on line 56. Both loops share the same alias variables to make the code easier to read.

The value of μ is calculated in an interesting fashion. First the value $\hat{W}_{ix}$ is reduced modulo β and cast to a mp_digit. This forces the compiler to use a single precision multiplication and prevents any concerns about loss of precision. Line 111 fixes the carry for the next iteration of the loop by propagating the carry from $\hat{W}_{ix}$ to $\hat{W}_{ix+1}$.

The for loop on line 110 propagates the rest of the carries upwards through the columns. The for loop on line 127 reduces the columns modulo β and shifts them k places at the same time. The alias $_\hat{W}$ actually refers to the array $\hat{W}$ starting at the $n.used$'th digit, that is $_\hat{W}_t = \hat{W}_{n.used+t}$.

6.3.4 Montgomery Setup

To calculate the variable ρ a relatively simple algorithm will be required.

Algorithm **mp_montgomery_setup**.
Input. mp_int n ($n > 1$ and $(n, 2) = 1$)
Output. $\rho \equiv -1/n_0 \pmod{\beta}$

1. $b \leftarrow n_0$
2. If b is even return(MP_VAL)
3. $x \leftarrow (((b + 2)$ AND $4) << 1) + b$
4. for k from 0 to $\lceil lg(lg(\beta)) \rceil - 2$ do
   4.1 $x \leftarrow x \cdot (2 - bx)$
5. $\rho \leftarrow \beta - x \pmod{\beta}$
6. Return(MP_OKAY).

Figure 6.11: Algorithm mp_montgomery_setup

Algorithm mp_montgomery_setup. This algorithm will calculate the value of ρ required within the Montgomery reduction algorithms. It uses a very interesting trick to calculate $1/n_0$ when β is a power of two.

File: bn_mp_montgomery_setup.c
```
001   #ifdef BN_MP_MONTGOMERY_SETUP_C
...
```

This source code computes the value of ρ required to perform Montgomery reduction. It has been modified to avoid performing excess multiplications when β is not the default 28-bits.

6.4 The Diminished Radix Algorithm

The Diminished Radix method of modular reduction [9] is a fairly clever technique which can be more efficient than either the Barrett or Montgomery methods for certain forms of moduli. The technique is based on the following simple congruence.

$$(x \bmod n) + k\lfloor x/n \rfloor \equiv x \pmod{(n - k)} \tag{6.6}$$

This observation was used in the MMB [10] block cipher to create a diffusion primitive. It used the fact that if $n = 2^{31}$ and $k = 1$ that then a x86 multiplier

could produce the 62-bit product and use the "shrd" instruction to perform a double-precision right shift. The proof of the above equation is very simple. First write x in the product form.

$$x = qn + r \tag{6.7}$$

Now reduce both sides modulo $(n - k)$.

$$x \equiv qk + r \pmod{(n-k)} \tag{6.8}$$

The variable n reduces modulo $n - k$ to k. By putting $q = \lfloor x/n \rfloor$ and $r = x \bmod n$ into the equation the original congruence is reproduced, thus concluding the proof. The following algorithm is based on this observation.

Algorithm **Diminished Radix Reduction**.
Input. Integer x, n, k
Output. $x \bmod (n - k)$

1. $q \leftarrow \lfloor x/n \rfloor$
2. $q \leftarrow k \cdot q$
3. $x \leftarrow x \pmod{n}$
4. $x \leftarrow x + q$
5. If $x \geq (n - k)$ then
   5.1 $x \leftarrow x - (n - k)$
   5.2 Goto step 1.
6. Return x

Figure 6.12: Algorithm Diminished Radix Reduction

This algorithm will reduce x modulo $n - k$ and return the residue. If $0 \leq x < (n - k)^2$ then the algorithm will loop almost always once or twice and occasionally three times. For simplicity sake the value of x is bounded by the following simple polynomial.

$$0 \leq x < n^2 + k^2 - 2nk \tag{6.9}$$

The true bound is $0 \leq x < (n-k-1)^2$ but this has quite a few more terms. The value of q after step 1 is bounded by the following.

$$q < n - 2k - k^2/n \tag{6.10}$$

$x = 123456789, n = 256, k = 3$
$q \leftarrow \lfloor x/n \rfloor = 482253$
$q \leftarrow q * k = 1446759$
$x \leftarrow x \bmod n = 21$
$x \leftarrow x + q = 1446780$
$x \leftarrow x - (n - k) = 1446527$
$q \leftarrow \lfloor x/n \rfloor = 5650$
$q \leftarrow q * k = 16950$
$x \leftarrow x \bmod n = 127$
$x \leftarrow x + q = 17077$
$x \leftarrow x - (n - k) = 16824$
$q \leftarrow \lfloor x/n \rfloor = 65$
$q \leftarrow q * k = 195$
$x \leftarrow x \bmod n = 184$
$x \leftarrow x + q = 379$
$x \leftarrow x - (n - k) = 126$

Figure 6.13: Example Diminished Radix Reduction

Since k^2 is going to be considerably smaller than n that term will always be zero. The value of x after step 3 is bounded trivially as $0 \leq x < n$. By step four the sum $x + q$ is bounded by

$$0 \leq q + x < (k + 1)n - 2k^2 - 1 \tag{6.11}$$

With a second pass q will be loosely bounded by $0 \leq q < k^2$ after step 2 while x will still be loosely bounded by $0 \leq x < n$ after step 3. After the second pass it is highly unlike that the sum in step 4 will exceed $n - k$. In practice fewer than three passes of the algorithm are required to reduce virtually every input in the range $0 \leq x < (n - k - 1)^2$.

Figure 6.13 demonstrates the reduction of $x = 123456789$ modulo $n-k = 253$ when $n = 256$ and $k = 3$. Note that even while x is considerably larger than $(n - k - 1)^2 = 63504$ the algorithm still converges on the modular residue exceedingly fast. In this case only three passes were required to find the residue $x \equiv 126$.

6.4.1 Choice of Moduli

On the surface this algorithm looks like a very expensive algorithm. It requires a couple of subtractions followed by multiplication and other modular reductions. The usefulness of this algorithm becomes exceedingly clear when an appropriate modulus is chosen.

Division in general is a very expensive operation to perform. The one exception is when the division is by a power of the radix of representation used. Division by ten for example is simple for pencil and paper mathematics since it amounts to shifting the decimal place to the right. Similarly division by two (*or powers of two*) is very simple for binary computers to perform. It would therefore seem logical to choose n of the form 2^p which would imply that $\lfloor x/n \rfloor$ is a simple shift of x right p bits.

However, there is one operation related to division of power of twos that is even faster than this. If $n = \beta^p$ then the division may be performed by moving whole digits to the right p places. In practice division by β^p is much faster than division by 2^p for any p. Also with the choice of $n = \beta^p$ reducing x modulo n merely requires zeroing the digits above the $p - 1$'th digit of x.

Throughout the next section the term "restricted modulus" will refer to a modulus of the form $\beta^p - k$ whereas the term "unrestricted modulus" will refer to a modulus of the form $2^p - k$. The word "restricted" in this case refers to the fact that it is based on the 2^p logic except p must be a multiple of $lg(\beta)$.

6.4.2 Choice of k

Now that division and reduction (*step 1 and 3 of figure 6.12*) have been optimized to simple digit operations the multiplication by k in step 2 is the most expensive operation. Fortunately the choice of k is not terribly limited. For all intents and purposes it might as well be a single digit. The smaller the value of k is the faster the algorithm will be.

6.4.3 Restricted Diminished Radix Reduction

The restricted Diminished Radix algorithm can quickly reduce an input modulo a modulus of the form $n = \beta^p - k$. This algorithm can reduce an input x within the range $0 \le x < n^2$ using only a couple passes of the algorithm demonstrated in figure 6.12. The implementation of this algorithm has been optimized to avoid additional overhead associated with a division by β^p, the multiplication by k or the addition of x and q. The resulting algorithm is very efficient and

can lead to substantial improvements over Barrett and Montgomery reduction when modular exponentiations are performed.

Algorithm **mp_dr_reduce**.
Input. mp_int x, n and a mp_digit $k = \beta - n_0$
$(0 \le x < n^2,\ n > 1,\ 0 < k < \beta)$
Output. $x \bmod n$

1. $m \leftarrow n.used$
2. If $x.alloc < 2m$ then grow x to $2m$ digits.
3. $\mu \leftarrow 0$
4. for i from 0 to $m - 1$ do
   4.1 $\hat{r} \leftarrow k \cdot x_{m+i} + x_i + \mu$
   4.2 $x_i \leftarrow \hat{r} \pmod{\beta}$
   4.3 $\mu \leftarrow \lfloor \hat{r}/\beta \rfloor$
5. $x_m \leftarrow \mu$
6. for i from $m + 1$ to $x.used - 1$ do
   6.1 $x_i \leftarrow 0$
7. Clamp excess digits of x.
8. If $x \ge n$ then
   8.1 $x \leftarrow x - n$
   8.2 Goto step 3.
9. Return(MP_OKAY).

Figure 6.14: Algorithm mp_dr_reduce

Algorithm mp_dr_reduce. This algorithm will perform the Dimished Radix reduction of x modulo n. It has similar restrictions to that of the Barrett reduction with the addition that n must be of the form $n = \beta^m - k$ where $0 < k < \beta$.

This algorithm essentially implements the pseudo-code in figure 6.12 except with a slight optimization. The division by β^m, multiplication by k and addition of $x \bmod \beta^m$ are all performed simultaneously inside the loop on step 4. The division by β^m is emulated by accessing the term at the $m+i$'th position which is subsequently multiplied by k and added to the term at the i'th position. After the loop the m'th digit is set to the carry and the upper digits are zeroed. Steps 5 and 6 emulate the reduction modulo β^m that should have happend to x before the addition of the multiple of the upper half.

At step 8 if x is still larger than n another pass of the algorithm is required. First n is subtracted from x and then the algorithm resumes at step 3.

File: bn_mp_dr_reduce.c
```
001   #ifdef BN_MP_DR_REDUCE_C
```
...

The first step is to grow x as required to $2m$ digits since the reduction is performed in place on x. The label on line 48 is where the algorithm will resume if further reduction passes are required. In theory it could be placed at the top of the function however, the size of the modulus and question of whether x is large enough are invariant after the first pass meaning that it would be a waste of time.

The aliases $tmpx1$ and $tmpx2$ refer to the digits of x where the latter is offset by m digits. By reading digits from x offset by m digits a division by β^m can be simulated virtually for free. The loop on line 60 performs the bulk of the work (*corresponds to step 4 of algorithm 7.11*) in this algorithm.

By line 67 the pointer $tmpx1$ points to the m'th digit of x which is where the final carry will be placed. Similarly by line 70 the same pointer will point to the $m + 1$'th digit where the zeroes will be placed.

Since the algorithm is only valid if both x and n are greater than zero an unsigned comparison suffices to determine if another pass is required. With the same logic at line 81 the value of x is known to be greater than or equal to n meaning that an unsigned subtraction can be used as well. Since the destination of the subtraction is the larger of the inputs the call to algorithm s_mp_sub cannot fail and the return code does not need to be checked.

Setup

To setup the restricted Diminished Radix algorithm the value $k = \beta - n_0$ is required. This algorithm is not really complicated but provided for completeness.

Algorithm **mp_dr_setup**.
Input. mp_int n
Output. $k = \beta - n_0$
1. $k \leftarrow \beta - n_0$

Figure 6.15: Algorithm mp_dr_setup

File: bn_mp_dr_setup.c

```
001   #ifdef BN_MP_DR_SETUP_C
...
```

Modulus Detection

Another algorithm which will be useful is the ability to detect a restricted Diminished Radix modulus. An integer is said to be of restricted Diminished Radix form if all of the digits are equal to $\beta - 1$ except the trailing digit which may be any value.

Algorithm **mp_dr_is_modulus**.
Input. mp_int n
Output. 1 if n is in D.R form, 0 otherwise
1. If $n.used < 2$ then return(0).
2. for ix from 1 to $n.used - 1$ do
2.1 If $n_{ix} \neq \beta - 1$ return(0).
3. Return(1).

Figure 6.16: Algorithm mp_dr_is_modulus

Algorithm mp_dr_is_modulus. This algorithm determines if a value is in Diminished Radix form. Step 1 rejects obvious cases where fewer than two digits are in the mp_int. Step 2 tests all but the first digit to see if they are equal to $\beta - 1$. If the algorithm manages to get to step 3 then n must be of Diminished Radix form.

File: bn_mp_dr_is_modulus.c

```
001   #ifdef BN_MP_DR_IS_MODULUS_C
...
```

6.4.4 Unrestricted Diminished Radix Reduction

The unrestricted Diminished Radix algorithm allows modular reductions to be performed when the modulus is of the form $2^p - k$. This algorithm is a straightforward adaptation of algorithm 6.12.

In general the restricted Diminished Radix reduction algorithm is much faster since it has considerably lower overhead. However, this new algorithm is much faster than either Montgomery or Barrett reduction when the moduli are of the appropriate form.

Algorithm mp_reduce_2k. This algorithm quickly reduces an input a modulo an unrestricted Diminished Radix modulus n. Division by 2^p is emulated with a right shift which makes the algorithm fairly inexpensive to use.

File: bn_mp_reduce_2k.c

Algorithm **mp_reduce_2k**.
Input. mp_int a and n. mp_digit k
($a \ge 0$, $n > 1$, $0 < k < \beta$, $n + k$ is a power of two)
Output. $a \pmod n$

1. $p \leftarrow \lceil lg(n) \rceil$ (*mp_count_bits*)
2. While $a \ge n$ do
   2.1 $q \leftarrow \lfloor a/2^p \rfloor$ (*mp_div_2d*)
   2.2 $a \leftarrow a \pmod{2^p}$ (*mp_mod_2d*)
   2.3 $q \leftarrow q \cdot k$ (*mp_mul_d*)
   2.4 $a \leftarrow a - q$ (*s_mp_sub*)
   2.5 If $a \ge n$ then do
      2.5.1 $a \leftarrow a - n$
3. Return(*MP_OKAY*).

Figure 6.17: Algorithm mp_reduce_2k

```
001   #ifdef BN_MP_REDUCE_2K_C
...
```

The algorithm mp_count_bits calculates the number of bits in an mp_int which is used to find the initial value of p. The call to mp_div_2d on line 28 calculates both the quotient q and the remainder a required. By doing both in a single function call the code size is kept fairly small. The multiplication by k is only performed if $k > 1$. This allows reductions modulo $2^p - 1$ to be performed without any multiplications.

The unsigned s_mp_add, mp_cmp_mag and s_mp_sub are used in place of their full sign counterparts since the inputs are only valid if they are positive. By using the unsigned versions the overhead is kept to a minimum.

Unrestricted Setup

To setup this reduction algorithm the value of $k = 2^p - n$ is required.

Algorithm mp_reduce_2k_setup. This algorithm computes the value of k required for the algorithm mp_reduce_2k. By making a temporary variable x equal to 2^p a subtraction is sufficient to solve for k. Alternatively if n has more than one digit the value of k is simply $\beta - n_0$.

File: bn_mp_reduce_2k_setup.c

```
001   #ifdef BN_MP_REDUCE_2K_SETUP_C
...
```

Algorithm **mp_reduce_2k_setup**.
Input. mp_int n
Output. $k = 2^p - n$
1. $p \leftarrow \lceil lg(n) \rceil$ (mp_count_bits)
2. $x \leftarrow 2^p$ (mp_2expt)
3. $x \leftarrow x - n$ (mp_sub)
4. $k \leftarrow x_0$
5. Return(MP_OKAY).

Figure 6.18: Algorithm mp_reduce_2k_setup

Unrestricted Detection

An integer n is a valid unrestricted Diminished Radix modulus if either of the following are true.

1. The number has only one digit.
2. The number has more than one digit and every bit from the β'th to the most significant is one.

If either condition is true than there is a power of two 2^p such that $0 < 2^p - n < \beta$. If the input is only one digit than it will always be of the correct form. Otherwise all of the bits above the first digit must be one. This arises from the fact that there will be value of k that when added to the modulus causes a carry in the first digit which propagates all the way to the most significant bit. The resulting sum will be a power of two.

Algorithm **mp_reduce_is_2k**.
Input. mp_int n
Output. 1 if of proper form, 0 otherwise
1. If $n.used = 0$ then return(0).
2. If $n.used = 1$ then return(1).
3. $p \leftarrow \lceil lg(n) \rceil$ (mp_count_bits)
4. for x from $lg(\beta)$ to p do
4.1 If the (x mod $lg(\beta)$)'th bit of the $\lfloor x/lg(\beta) \rfloor$ of n is zero then return(0).
5. Return(1).

Figure 6.19: Algorithm mp_reduce_is_2k

Algorithm mp_reduce_is_2k. This algorithm quickly determines if a modulus is of the form required for algorithm mp_reduce_2k to function properly.

File: bn_mp_reduce_is_2k.c

```
001   #ifdef BN_MP_REDUCE_IS_2K_C
...
```

6.5 Algorithm Comparison

So far three very different algorithms for modular reduction have been discussed. Each of the algorithms have their own strengths and weaknesses that makes having such a selection very useful. The following table sumarizes the three algorithms along with comparisons of work factors. Since all three algorithms have the restriction that $0 \le x < n^2$ and $n > 1$ those limitations are not included in the table.

Method	**Work Required**	**Limitations**	$m = 8$	$m = 32$	$m = 64$
Barrett	$m^2 + 2m - 1$	None	79	1087	4223
Montgomery	$m^2 + m$	n must be odd	72	1056	4160
D.R.	$2m$	$n = \beta^m - k$	16	64	128

In theory Montgomery and Barrett reductions would require roughly the same amount of time to complete. However, in practice since Montgomery reduction can be written as a single function with the Comba technique it is much faster. Barrett reduction suffers from the overhead of calling the half precision multipliers, addition and division by β algorithms.

For almost every cryptographic algorithm Montgomery reduction is the algorithm of choice. The one set of algorithms where Diminished Radix reduction truly shines are based on the discrete logarithm problem such as Diffie-Hellman [12] and ElGamal [21]. In these algorithms primes of the form $\beta^m - k$ can be found and shared amongst users. These primes will allow the Diminished Radix algorithm to be used in modular exponentiation to greatly speed up the operation.

Exercises

[3] Prove that the "trick" in algorithm mp_montgomery_setup actually calculates the correct value of ρ.

[2] Devise an algorithm to reduce modulo $n + k$ for small k quickly.

[4] Prove that the pseudo-code algorithm "Diminished Radix Reduction" (*figure 6.12*) terminates. Also prove the probability that it will terminate within $1 \le k \le 10$ iterations.

Chapter 7

Exponentiation

Exponentiation is the operation of raising one variable to the power of another, for example, a^b. A variant of exponentiation, computed in a finite field or ring, is called modular exponentiation. This latter style of operation is typically used in public key cryptosystems such as RSA and Diffie-Hellman. The ability to quickly compute modular exponentiations is of great benefit to any such cryptosystem and many methods have been sought to speed it up.

7.1 Exponentiation Basics

A trivial algorithm would simply multiply a against itself $b-1$ times to compute the exponentiation desired. However, as b grows in size the number of multiplications becomes prohibitive. Imagine what would happen if $b \sim 2^{1024}$ as is the case when computing an RSA signature with a 1024-bit key. Such a calculation could never be completed as it would take simply far too long.

Fortunately there is a very simple algorithm based on the laws of exponents. Recall that $lg_a(a^b) = b$ and that $lg_a(a^b a^c) = b + c$ which are two trivial relationships between the base and the exponent. Let b_i represent the i'th bit of b starting from the least significant bit. If b is a k-bit integer than the following equation is true.

$$a^b = \prod_{i=0}^{k-1} a^{2^i \cdot b_i} \tag{7.1}$$

By taking the base a logarithm of both sides of the equation the following equation is the result.

$$b = \sum_{i=0}^{k-1} 2^i \cdot b_i \tag{7.2}$$

The term a^{2^i} can be found from the $i-1$'th term by squaring the term since $\left(a^{2^i}\right)^2$ is equal to $a^{2^{i+1}}$. This observation forms the basis of essentially all fast exponentiation algorithms. It requires k squarings and on average $\frac{k}{2}$ multiplications to compute the result. This is indeed quite an improvement over simply multiplying by a a total of $b-1$ times.

While this current method is a considerable speed up there are further improvements to be made. For example, the a^{2^i} term does not need to be computed in an auxilary variable. Consider the following equivalent algorithm.

Algorithm **Left to Right Exponentiation**.
Input. Integer a, b and k
Output. $c = a^b$

1. $c \leftarrow 1$
2. for i from $k-1$ to 0 do
   2.1 $c \leftarrow c^2$
   2.2 $c \leftarrow c \cdot a^{b_i}$
3. Return c.

Figure 7.1: Left to Right Exponentiation

This algorithm starts from the most significant bit and works towards the least significant bit. When the i'th bit of b is set a is multiplied against the current product. In each iteration the product is squared which doubles the exponent of the individual terms of the product.

For example, let $b = 101100_2 \equiv 44_{10}$. The following chart demonstrates the actions of the algorithm.

Value of i	**Value of c**
-	1
5	a
4	a^2
3	$a^4 \cdot a$
2	$a^8 \cdot a^2 \cdot a$
1	$a^{16} \cdot a^4 \cdot a^2$
0	$a^{32} \cdot a^8 \cdot a^4$

Figure 7.2: Example of Left to Right Exponentiation

When the product $a^{32} \cdot a^8 \cdot a^4$ is simplified it is equal a^{44} which is the desired exponentiation. This particular algorithm is called "Left to Right" because it reads the exponent in that order. All of the exponentiation algorithms that will be presented are of this nature.

7.1.1 Single Digit Exponentiation

The first algorithm in the series of exponentiation algorithms will be an unbounded algorithm where the exponent is a single digit. It is intended to be used when a small power of an input is required (*e.g.* a^5). It is faster than simply multiplying $b-1$ times for all values of b that are greater than three.

Algorithm **mp_expt_d**.
Input. mp_int a and mp_digit b
Output. $c = a^b$

1. $g \leftarrow a$ (mp_init_copy)
2. $c \leftarrow 1$ (mp_set)
3. for x from 1 to $lg(\beta)$ do
   3.1 $c \leftarrow c^2$ (mp_sqr)
   3.2 If b AND $2^{lg(\beta)-1} \neq 0$ then
      3.2.1 $c \leftarrow c \cdot g$ (mp_mul)
   3.3 $b \leftarrow b << 1$
4. Clear g.
5. Return(MP_OKAY).

Figure 7.3: Algorithm mp_expt_d

Algorithm mp_expt_d. This algorithm computes the value of a raised to the power of a single digit b. It uses the left to right exponentiation algorithm to quickly compute the exponentiation. It is loosely based on algorithm 14.79 of HAC [2, pp. 615] with the difference that the exponent is a fixed width.

A copy of a is made first to allow destination variable c be the same as the source variable a. The result is set to the initial value of 1 in the subsequent step.

Inside the loop the exponent is read from the most significant bit first down to the least significant bit. First c is invariably squared on step 3.1. In the following step if the most significant bit of b is one the copy of a is multiplied against c. The value of b is shifted left one bit to make the next bit down from the most signficant bit the new most significant bit. In effect each iteration of the loop moves the bits of the exponent b upwards to the most significant location.

File: bn_mp_expt_d_ex.c
```
001   #ifdef BN_MP_EXPT_D_EX_C
...
```

This describes only the algorithm that is used when the parameter $fast$ is 0. Line 28 sets the initial value of the result to 1. Next the loop on line 52 steps through each bit of the exponent starting from the most significant down towards the least significant. The invariant squaring operation placed on line 54 is performed first. After the squaring the result c is multiplied by the base

g if and only if the most significant bit of the exponent is set. The shift on line 68 moves all of the bits of the exponent upwards towards the most significant location.

7.2 k-ary Exponentiation

When calculating an exponentiation the most time consuming bottleneck is the multiplications which are in general a small factor slower than squaring. Recall from the previous algorithm that b_i refers to the i'th bit of the exponent b. Suppose instead it referred to the i'th k-bit digit of the exponent of b. For $k = 1$ the definitions are synonymous and for $k > 1$ algorithm 7.4 computes the same exponentiation. A group of k bits from the exponent is called a *window*. That is it is a small window on only a portion of the entire exponent. Consider the following modification to the basic left to right exponentiation algorithm.

Algorithm **k-ary Exponentiation**.
Input. Integer a, b, k and t
Output. $c = a^b$

1. $c \leftarrow 1$
2. for i from $t - 1$ to 0 do
   2.1 $c \leftarrow c^{2^k}$
   2.2 Extract the i'th k-bit word from b and store it in g.
   2.3 $c \leftarrow c \cdot a^g$
3. Return c.

Figure 7.4: k-ary Exponentiation

The squaring on step 2.1 can be calculated by squaring the value c successively k times. If the values of a^g for $0 < g < 2^k$ have been precomputed this algorithm requires only t multiplications and tk squarings. The table can be generated with $2^{k-1} - 1$ squarings and $2^{k-1} + 1$ multiplications. This algorithm assumes that the number of bits in the exponent is evenly divisible by k. However, when it is not the remaining $0 < x \leq k - 1$ bits can be handled with algorithm 7.1.

Suppose $k = 4$ and $t = 100$. This modified algorithm will require 109 multiplications and 408 squarings to compute the exponentiation. The original algorithm would on average have required 200 multiplications and 400 squrings

to compute the same value. The total number of squarings has increased slightly but the number of multiplications has nearly halved.

7.2.1 Optimal Values of k

An optimal value of k will minimize $2^k + \lceil n/k \rceil + n - 1$ for a fixed number of bits in the exponent n. The simplest approach is to brute force search amongst the values $k = 2, 3, \ldots, 8$ for the lowest result. Table 7.5 lists optimal values of k for various exponent sizes and compares the number of multiplication and squarings required against algorithm 7.1.

Exponent (bits)	**Optimal k**	**Work at k**	**Work with 7.1**
16	2	27	24
32	3	49	48
64	3	92	96
128	4	175	192
256	4	335	384
512	5	645	768
1024	6	1257	1536
2048	6	2452	3072
4096	7	4808	6144

Figure 7.5: Optimal Values of k for k-ary Exponentiation

7.2.2 Sliding-Window Exponentiation

A simple modification to the previous algorithm is only generate the upper half of the table in the range $2^{k-1} \leq g < 2^k$. Essentially this is a table for all values of g where the most significant bit of g is a one. However, in order for this to be allowed in the algorithm values of g in the range $0 \leq g < 2^{k-1}$ must be avoided.

Table 7.6 lists optimal values of k for various exponent sizes and compares the work required against algorithm 7.4.

Exponent (bits)	Optimal k	Work at k	Work with 7.4
16	3	24	27
32	3	45	49
64	4	87	92
128	4	167	175
256	5	322	335
512	6	628	645
1024	6	1225	1257
2048	7	2403	2452
4096	8	4735	4808

Figure 7.6: Optimal Values of k for Sliding Window Exponentiation

Algorithm **Sliding Window k-ary Exponentiation**.
Input. Integer a, b, k and t
Output. $c = a^b$

1. $c \leftarrow 1$
2. for i from $t-1$ to 0 do
   2.1 If the i'th bit of b is a zero then
      2.1.1 $c \leftarrow c^2$
   2.2 else do
      2.2.1 $c \leftarrow c^{2^k}$
      2.2.2 Extract the k bits from $(b_i b_{i-1} \ldots b_{i-(k-1)})$ and store it in g.
      2.2.3 $c \leftarrow c \cdot a^g$
      2.2.4 $i \leftarrow i - k$
3. Return c.

Figure 7.7: Sliding Window k-ary Exponentiation

Similar to the previous algorithm this algorithm must have a special handler when fewer than k bits are left in the exponent. While this algorithm requires the same number of squarings it can potentially have fewer multiplications. The pre-computed table a^g is also half the size as the previous table.

Consider the exponent $b = 111101011001000_2 \equiv 31432_{10}$ with $k = 3$ using both algorithms. The first algorithm will divide the exponent up as the following five 3-bit words $b \equiv (111, 101, 011, 001, 000)_2$. The second algorithm will break the exponent as $b \equiv (111, 101, 0, 110, 0, 100, 0)_2$. The single digit 0 in the second

representation are where a single squaring took place instead of a squaring and multiplication. In total the first method requires 10 multiplications and 18 squarings. The second method requires 8 multiplications and 18 squarings.

In general the sliding window method is never slower than the generic k-ary method and often it is slightly faster.

7.3 Modular Exponentiation

Modular exponentiation is essentially computing the power of a base within a finite field or ring. For example, computing $d \equiv a^b \pmod{c}$ is a modular exponentiation. Instead of first computing a^b and then reducing it modulo c the intermediate result is reduced modulo c after every squaring or multiplication operation.

This guarantees that any intermediate result is bounded by $0 \le d \le c^2 - 2c + 1$ and can be reduced modulo c quickly using one of the algorithms presented in chapter six.

Before the actual modular exponentiation algorithm can be written a wrapper algorithm must be written first. This algorithm will allow the exponent b to be negative which is computed as $c \equiv (1/a)^{|b|} \pmod{d}$. The value of $(1/a) \bmod c$ is computed using the modular inverse (*see 9.4*). If no inverse exists the algorithm terminates with an error.

Algorithm **mp_exptmod**.
Input. mp_int a, b and c
Output. $y \equiv g^x \pmod{p}$

1. If $c.sign = MP_NEG$ return(MP_VAL).
2. If $b.sign = MP_NEG$ then
   2.1 $g' \leftarrow g^{-1} \pmod{c}$
   2.2 $x' \leftarrow |x|$
   2.3 Compute $d \equiv g'^{x'} \pmod{c}$ via recursion.
3. if p is odd **OR** p is a D.R. modulus then
   3.1 Compute $y \equiv g^x \pmod{p}$ via algorithm mp_exptmod_fast.
4. else
   4.1 Compute $y \equiv g^x \pmod{p}$ via algorithm s_mp_exptmod.

Figure 7.8: Algorithm mp_exptmod

Algorithm mp_exptmod. The first algorithm which actually performs

modular exponentiation is algorithm s_mp_exptmod. It is a sliding window k-ary algorithm which uses Barrett reduction to reduce the product modulo p. The second algorithm mp_exptmod_fast performs the same operation except it uses either Montgomery or Diminished Radix reduction. The two latter reduction algorithms are clumped in the same exponentiation algorithm since their arguments are essentially the same (*two mp_ints and one mp_digit*).

File: bn_mp_exptmod.c

```
001   #ifdef BN_MP_EXPTMOD_C
...
031   #ifdef BN_MP_INVMOD_C
...
071   #ifdef BN_MP_DR_IS_MODULUS_C
...
079   #ifdef BN_MP_REDUCE_IS_2K_C
...
087   #ifdef BN_MP_EXPTMOD_FAST_C
...
092   #ifdef BN_S_MP_EXPTMOD_C
...
099   #ifdef BN_MP_EXPTMOD_FAST_C
...
```

In order to keep the algorithms in a known state the first step on line 30 is to reject any negative modulus as input. If the exponent is negative the algorithm tries to perform a modular exponentiation with the modular inverse of the base G. The temporary variable $tmpG$ is assigned the modular inverse of G and $tmpX$ is assigned the absolute value of X. The algorithm will recuse with these new values with a positive exponent.

If the exponent is positive the algorithm resumes the exponentiation. Line 74 determines if the modulus is of the restricted Diminished Radix form. If it is not line 67 attempts to determine if it is of a unrestricted Diminished Radix form. The integer dr will take on one of three values.

1. $dr = 0$ means that the modulus is not of either restricted or unrestricted Diminished Radix form.

2. $dr = 1$ means that the modulus is of restricted Diminished Radix form.

3. $dr = 2$ means that the modulus is of unrestricted Diminished Radix form.

Line 70 determines if the fast modular exponentiation algorithm can be used. It is allowed if $dr \neq 0$ or if the modulus is odd. Otherwise, the slower s_mp_exptmod algorithm is used which uses Barrett reduction.

7.3.1 Barrett Modular Exponentiation

Algorithm **s_mp_exptmod**.
Input. mp_int a, b and c
Output. $y \equiv g^x \pmod p$

1. $k \leftarrow lg(x)$
2. $winsize \leftarrow \begin{cases} 2 & \text{if } k \leq 7 \\ 3 & \text{if } 7 < k \leq 36 \\ 4 & \text{if } 36 < k \leq 140 \\ 5 & \text{if } 140 < k \leq 450 \\ 6 & \text{if } 450 < k \leq 1303 \\ 7 & \text{if } 1303 < k \leq 3529 \\ 8 & \text{if } 3529 < k \end{cases}$
3. Initialize $2^{winsize}$ mp_ints in an array named M and one mp_int named μ
4. Calculate the μ required for Barrett Reduction (*mp_reduce_setup*).
5. $M_1 \leftarrow g \pmod p$

Setup the table of small powers of g. First find $g^{2^{winsize}}$ and then all multiples of it.
6. $k \leftarrow 2^{winsize-1}$
7. $M_k \leftarrow M_1$
8. for ix from 0 to $winsize - 2$ do
   8.1 $M_k \leftarrow (M_k)^2$ (*mp_sqr*)
   8.2 $M_k \leftarrow M_k \pmod p$ (*mp_reduce*)
9. for ix from $2^{winsize-1} + 1$ to $2^{winsize} - 1$ do
   9.1 $M_{ix} \leftarrow M_{ix-1} \cdot M_1$ (*mp_mul*)
   9.2 $M_{ix} \leftarrow M_{ix} \pmod p$ (*mp_reduce*)
10. $res \leftarrow 1$

Start Sliding Window.
11. $mode \leftarrow 0, bitcnt \leftarrow 1, buf \leftarrow 0, digidx \leftarrow x.used - 1, bitcpy \leftarrow 0, bitbuf \leftarrow 0$
12. Loop
   12.1 $bitcnt \leftarrow bitcnt - 1$
   12.2 If $bitcnt = 0$ then do
      12.2.1 If $digidx = -1$ goto step 13.
      12.2.2 $buf \leftarrow x_{digidx}$
      12.2.3 $digidx \leftarrow digidx - 1$
      12.2.4 $bitcnt \leftarrow lg(\beta)$

Continued on next page.

Figure 7.9: Algorithm s_mp_exptmod

Algorithm **s_mp_exptmod** (*continued*).
Input. mp_int a, b and c
Output. $y \equiv g^x \pmod{p}$

12.3 $y \leftarrow (buf >> (lg(\beta) - 1))$ AND 1
12.4 $buf \leftarrow buf << 1$
12.5 if $mode = 0$ and $y = 0$ then goto step 12.
12.6 if $mode = 1$ and $y = 0$ then do
  12.6.1 $res \leftarrow res^2$
  12.6.2 $res \leftarrow res \pmod{p}$
  12.6.3 Goto step 12.
12.7 $bitcpy \leftarrow bitcpy + 1$
12.8 $bitbuf \leftarrow bitbuf + (y << (winsize - bitcpy))$
12.9 $mode \leftarrow 2$
12.10 If $bitcpy = winsize$ then do
  Window is full so perform the squarings and single multiplication.
  12.10.1 for ix from 0 to $winsize - 1$ do
    12.10.1.1 $res \leftarrow res^2$
    12.10.1.2 $res \leftarrow res \pmod{p}$
  12.10.2 $res \leftarrow res \cdot M_{bitbuf}$
  12.10.3 $res \leftarrow res \pmod{p}$
  Reset the window.
  12.10.4 $bitcpy \leftarrow 0, bitbuf \leftarrow 0, mode \leftarrow 1$

No more windows left. Check for residual bits of exponent.
13. If $mode = 2$ and $bitcpy > 0$ then do
  13.1 for ix form 0 to $bitcpy - 1$ do
    13.1.1 $res \leftarrow res^2$
    13.1.2 $res \leftarrow res \pmod{p}$
    13.1.3 $bitbuf \leftarrow bitbuf << 1$
    13.1.4 If $bitbuf$ AND $2^{winsize} \neq 0$ then do
      13.1.4.1 $res \leftarrow res \cdot M_1$
      13.1.4.2 $res \leftarrow res \pmod{p}$
14. $y \leftarrow res$
15. Clear res, mu and the M array.
16. Return(MP_OKAY).

Figure 7.10: Algorithm s_mp_exptmod (continued)

Algorithm s_mp_exptmod. This algorithm computes the x'th power of g modulo p and stores the result in y. It takes advantage of the Barrett reduction

algorithm to keep the product small throughout the algorithm.

The first two steps determine the optimal window size based on the number of bits in the exponent. The larger the exponent the larger the window size becomes. After a window size $winsize$ has been chosen an array of $2^{winsize}$ mp_int variables is allocated. This table will hold the values of $g^x \pmod p$ for $2^{winsize-1} \le x < 2^{winsize}$.

After the table is allocated the first power of g is found. Since $g \ge p$ is allowed it must be first reduced modulo p to make the rest of the algorithm more efficient. The first element of the table at $2^{winsize-1}$ is found by squaring M_1 successively $winsize - 2$ times. The rest of the table elements are found by multiplying the previous element by M_1 modulo p.

Now that the table is available the sliding window may begin. The following list describes the functions of all the variables in the window.

1. The variable *mode* dictates how the bits of the exponent are interpreted.
   (a) When $mode = 0$ the bits are ignored since no non-zero bit of the exponent has been seen yet. For example, if the exponent were simply 1 then there would be $lg(\beta) - 1$ zero bits before the first non-zero bit. In this case bits are ignored until a non-zero bit is found.
   (b) When $mode = 1$ a non-zero bit has been seen before and a new $winsize$-bit window has not been formed yet. In this mode leading 0 bits are read and a single squaring is performed. If a non-zero bit is read a new window is created.
   (c) When $mode = 2$ the algorithm is in the middle of forming a window and new bits are appended to the window from the most significant bit downwards.
2. The variable *bitcnt* indicates how many bits are left in the current digit of the exponent left to be read. When it reaches zero a new digit is fetched from the exponent.
3. The variable buf holds the currently read digit of the exponent.
4. The variable *digidx* is an index into the exponents digits. It starts at the leading digit $x.used - 1$ and moves towards the trailing digit.
5. The variable *bitcpy* indicates how many bits are in the currently formed window. When it reaches *winsize* the window is flushed and the appropriate operations performed.

6. The variable *bitbuf* holds the current bits of the window being formed.

All of step 12 is the window processing loop. It will iterate while there are digits available form the exponent to read. The first step inside this loop is to extract a new digit if no more bits are available in the current digit. If there are no bits left a new digit is read and if there are no digits left than the loop terminates.

After a digit is made available step 12.3 will extract the most significant bit of the current digit and move all other bits in the digit upwards. In effect the digit is read from most significant bit to least significant bit and since the digits are read from leading to trailing edges the entire exponent is read from most significant bit to least significant bit.

At step 12.5 if the *mode* and currently extracted bit y are both zero the bit is ignored and the next bit is read. This prevents the algorithm from having to perform trivial squaring and reduction operations before the first non-zero bit is read. Step 12.6 and 12.7-10 handle the two cases of $mode = 1$ and $mode = 2$ respectively.

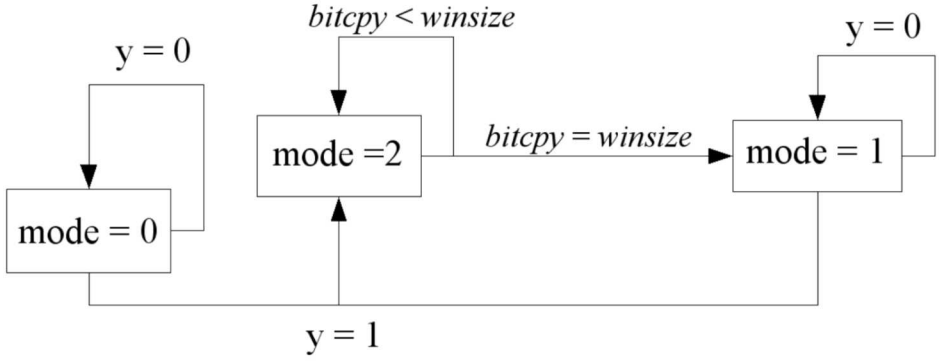


Figure 7.11: Sliding Window State Diagram

By step 13 there are no more digits left in the exponent. However, there may be partial bits in the window left. If $mode = 2$ then a Left-to-Right algorithm is used to process the remaining few bits.

File: bn_s_mp_exptmod.c

```
001   #ifdef BN_S_MP_EXPTMOD_C
...
```

Lines 32 through 44 determine the optimal window size based on the length of the exponent in bits. The window divisions are sorted from smallest to greatest so that in each **if** statement only one condition must be tested. For example, by the **if** statement on line 38 the value of x is already known to be greater than 140.

The conditional piece of code beginning on line 46 allows the window size to be restricted to five bits. This logic is used to ensure the table of precomputed powers of G remains relatively small.

The for loop on line 61 initializes the M array while lines 70 and 75 through 84 initialize the reduction function that will be used for this modulus.

– More later.

7.4 Quick Power of Two

Calculating $b = 2^a$ can be performed much quicker than with any of the previous algorithms. Recall that a logical shift left $m << k$ is equivalent to $m \cdot 2^k$. By this logic when $m = 1$ a quick power of two can be achieved.

Algorithm **mp_2expt**.
Input. integer b
Output. $a \leftarrow 2^b$

1. $a \leftarrow 0$
2. If $a.alloc < \lfloor b/lg(\beta) \rfloor + 1$ then grow a appropriately.
3. $a.used \leftarrow \lfloor b/lg(\beta) \rfloor + 1$
4. $a_{\lfloor b/lg(\beta) \rfloor} \leftarrow 1 << (b \bmod lg(\beta))$
5. Return(MP_OKAY).

Figure 7.12: Algorithm mp_2expt

Algorithm mp_2expt.

File: bn_mp_2expt.c

```
001   #ifdef BN_MP_2EXPT_C
...
```

Chapter 8

Higher Level Algorithms

This chapter discusses the various higher level algorithms that are required to complete a well rounded multiple precision integer package. These routines are less performance oriented than the algorithms of chapters five, six and seven but are no less important.

The first section describes a method of integer division with remainder that is universally well known. It provides the signed division logic for the package. The subsequent section discusses a set of algorithms which allow a single digit to be the 2nd operand for a variety of operations. These algorithms serve mostly to simplify other algorithms where small constants are required. The last two sections discuss how to manipulate various representations of integers. For example, converting from an mp_int to a string of character.

8.1 Integer Division with Remainder

Integer division aside from modular exponentiation is the most intensive algorithm to compute. Like addition, subtraction and multiplication the basis of this algorithm is the long-hand division algorithm taught to school children. Throughout this discussion several common variables will be used. Let x represent the divisor and y represent the dividend. Let q represent the integer quotient $\lfloor y/x \rfloor$ and let r represent the remainder $r = y - x\lfloor y/x \rfloor$. The following simple algorithm will be used to start the discussion.

Algorithm **Radix-β Integer Division**.
Input. integer x and y
Output. $q = \lfloor y/x \rfloor, r = y - xq$

1. $q \leftarrow 0$
2. $n \leftarrow ||y|| - ||x||$
3. for t from n down to 0 do
   3.1 Maximize k such that $kx\beta^t$ is less than or equal to y and $(k+1)x\beta^t$ is greater.
   3.2 $q \leftarrow q + k\beta^t$
   3.3 $y \leftarrow y - kx\beta^t$
4. $r \leftarrow y$
5. Return(q, r)

Figure 8.1: Algorithm Radix-β Integer Division

As children we are taught this very simple algorithm for the case of $\beta = 10$. Almost instinctively several optimizations are taught for which their reason of existing are never explained. For this example let $y = 5471$ represent the dividend and $x = 23$ represent the divisor.

To find the first digit of the quotient the value of k must be maximized such that $kx\beta^t$ is less than or equal to y and simultaneously $(k+1)x\beta^t$ is greater than y. Implicitly k is the maximum value the t'th digit of the quotient may have. The habitual method used to find the maximum is to "eyeball" the two numbers, typically only the leading digits and quickly estimate a quotient. By only using leading digits a much simpler division may be used to form an educated guess at what the value must be. In this case $k = \lfloor 54/23 \rfloor = 2$ quickly arises as a possible solution. Indeed $2x\beta^2 = 4600$ is less than $y = 5471$ and simultaneously $(k+1)x\beta^2 = 6900$ is larger than y. As a result $k\beta^2$ is added to the quotient which now equals $q = 200$ and 4600 is subtracted from y to give a remainder of $y = 841$.

Again this process is repeated to produce the quotient digit $k = 3$ which makes the quotient $q = 200 + 3\beta = 230$ and the remainder $y = 841 - 3x\beta = 181$. Finally the last iteration of the loop produces $k = 7$ which leads to the quotient $q = 230 + 7 = 237$ and the remainder $y = 181 - 7x = 20$. The final quotient and remainder found are $q = 237$ and $r = y = 20$ which are indeed correct since $237 \cdot 23 + 20 = 5471$ is true.

8.1.1 Quotient Estimation

As alluded to earlier the quotient digit k can be estimated from only the leading digits of both the divisor and dividend. When p leading digits are used from both the divisor and dividend to form an estimation the accuracy of the estimation rises as p grows. Technically speaking the estimation is based on assuming the lower $||y|| - p$ and $||x|| - p$ lower digits of the dividend and divisor are zero.

The value of the estimation may off by a few values in either direction and in general is fairly correct. A simplification [1, pp. 271] of the estimation technique is to use $t + 1$ digits of the dividend and t digits of the divisor, in particularly when $t = 1$. The estimate using this technique is never too small. For the following proof let $t = ||y|| - 1$ and $s = ||x|| - 1$ represent the most significant digits of the dividend and divisor respectively.

Proof. *The quotient* $\hat{k} = \lfloor (y_t\beta + y_{t-1})/x_s \rfloor$ *is greater than or equal to* $k = \lfloor y/(x \cdot \beta^{||y||-||x||-1}) \rfloor$. The first obvious case is when $\hat{k} = \beta - 1$ in which case the proof is concluded since the real quotient cannot be larger. For all other cases $\hat{k} = \lfloor (y_t\beta + y_{t-1})/x_s \rfloor$ and $\hat{k}x_s \geq y_t\beta + y_{t-1} - x_s + 1$. The latter portion of the inequalility $-x_s + 1$ arises from the fact that a truncated integer division will give the same quotient for at most $x_s - 1$ values. Next a series of inequalities will prove the hypothesis.

$$y - \hat{k}x \leq y - \hat{k}x_s\beta^s \tag{8.1}$$

This is trivially true since $x \geq x_s\beta^s$. Next we replace $\hat{k}x_s\beta^s$ by the previous inequality for $\hat{k}x_s$.

$$y - \hat{k}x \leq y_t\beta^t + \ldots + y_0 - (y_t\beta^t + y_{t-1}\beta^{t-1} - x_s\beta^t + \beta^s) \tag{8.2}$$

By simplifying the previous inequality the following inequality is formed.

$$y - \hat{k}x \leq y_{t-2}\beta^{t-2} + \ldots + y_0 + x_s\beta^s - \beta^s \tag{8.3}$$

Subsequently,

$$y_{t-2}\beta^{t-2} + \ldots + y_0 + x_s\beta^s - \beta^s < x_s\beta^s \leq x \tag{8.4}$$

Which proves that $y - \hat{k}x \leq x$ and by consequence $\hat{k} \geq k$ which concludes the proof. **QED**

8.1.2 Normalized Integers

For the purposes of division a normalized input is when the divisors leading digit x_n is greater than or equal to $\beta/2$. By multiplying both x and y by $j = \lfloor(\beta/2)/x_n\rfloor$ the quotient remains unchanged and the remainder is simply j times the original remainder. The purpose of normalization is to ensure the leading digit of the divisor is sufficiently large such that the estimated quotient will lie in the domain of a single digit. Consider the maximum dividend $(\beta-1)\cdot\beta+(\beta-1)$ and the minimum divisor $\beta/2$.

$$\frac{\beta^2-1}{\beta/2} \leq 2\beta - \frac{2}{\beta} \tag{8.5}$$

At most the quotient approaches 2β, however, in practice this will not occur since that would imply the previous quotient digit was too small.

8.1.3 Radix-β Division with Remainder

Algorithm **mp_div**.
Input. mp_int a, b
Output. $c = \lfloor a/b \rfloor$, $d = a - bc$

1. If $b = 0$ return(MP_VAL).
2. If $|a| < |b|$ then do
   2.1 $d \leftarrow a$
   2.2 $c \leftarrow 0$
   2.3 Return(MP_OKAY).

Setup the quotient to receive the digits.
3. Grow q to $a.used + 2$ digits.
4. $q \leftarrow 0$
5. $x \leftarrow |a|, y \leftarrow |b|$
6. $sign \leftarrow \begin{cases} MP_ZPOS & \text{if } a.sign = b.sign \\ MP_NEG & \text{otherwise} \end{cases}$

Normalize the inputs such that the leading digit of y is greater than or equal to $\beta/2$.
7. $norm \leftarrow (lg(\beta) - 1) - (\lceil lg(y) \rceil \pmod{lg(\beta)})$
8. $x \leftarrow x \cdot 2^{norm}, y \leftarrow y \cdot 2^{norm}$

Find the leading digit of the quotient.
9. $n \leftarrow x.used - 1, t \leftarrow y.used - 1$
10. $y \leftarrow y \cdot \beta^{n-t}$
11. While $(x \geq y)$ do
    11.1 $q_{n-t} \leftarrow q_{n-t} + 1$
    11.2 $x \leftarrow x - y$
12. $y \leftarrow \lfloor y/\beta^{n-t} \rfloor$

Continued on the next page.

Figure 8.2: Algorithm mp_div

Algorithm **mp_div** (continued).
Input. mp_int a, b
Output. $c = \lfloor a/b \rfloor$, $d = a - bc$

Now find the remainder fo the digits.
13. for i from n down to $(t + 1)$ do
  13.1 If $i > x.used$ then jump to the next iteration of this loop.
  13.2 If $x_i = y_t$ then
    13.2.1 $q_{i-t-1} \leftarrow \beta - 1$
  13.3 else
    13.3.1 $\hat{r} \leftarrow x_i \cdot \beta + x_{i-1}$
    13.3.2 $\hat{r} \leftarrow \lfloor \hat{r}/y_t \rfloor$
    13.3.3 $q_{i-t-1} \leftarrow \hat{r}$
  13.4 $q_{i-t-1} \leftarrow q_{i-t-1} + 1$

Fixup quotient estimation.
  13.5 Loop
    13.5.1 $q_{i-t-1} \leftarrow q_{i-t-1} - 1$
    13.5.2 t1 $\leftarrow 0$
    13.5.3 $\text{t1}_0 \leftarrow y_{t-1}$, $\text{t1}_1 \leftarrow y_t$, $\text{t1}.used \leftarrow 2$
    13.5.4 $t1 \leftarrow t1 \cdot q_{i-t-1}$
    13.5.5 $\text{t2}_0 \leftarrow x_{i-2}$, $\text{t2}_1 \leftarrow x_{i-1}$, $\text{t2}_2 \leftarrow x_i$, $\text{t2}.used \leftarrow 3$
    13.5.6 If $|t1| > |t2|$ then goto step 13.5.
  13.6 t1 $\leftarrow y \cdot q_{i-t-1}$
  13.7 t1 $\leftarrow$ t1 $\cdot \beta^{i-t-1}$
  13.8 $x \leftarrow x-$ t1
  13.9 If $x.sign = MP_NEG$ then
    13.10 t1 $\leftarrow y$
    13.11 t1 $\leftarrow$ t1 $\cdot \beta^{i-t-1}$
    13.12 $x \leftarrow x+$ t1
    13.13 $q_{i-t-1} \leftarrow q_{i-t-1} - 1$

Finalize the result.
14. Clamp excess digits of q
15. $c \leftarrow q, c.sign \leftarrow sign$
16. $x.sign \leftarrow a.sign$
17. $d \leftarrow \lfloor x/2^{norm} \rfloor$
18. Return(MP_OKAY).

Figure 8.3: Algorithm mp_div (continued)

Algorithm mp_div. This algorithm will calculate quotient and remainder from an integer division given a dividend and divisor. The algorithm is a signed division and will produce a fully qualified quotient and remainder.

First the divisor b must be non-zero which is enforced in step one. If the divisor is larger than the dividend than the quotient is implicitly zero and the remainder is the dividend.

After the first two trivial cases of inputs are handled the variable q is setup to receive the digits of the quotient. Two unsigned copies of the divisor y and dividend x are made as well. The core of the division algorithm is an unsigned division and will only work if the values are positive. Now the two values x and y must be normalized such that the leading digit of y is greater than or equal to $\beta/2$. This is performed by shifting both to the left by enough bits to get the desired normalization.

At this point the division algorithm can begin producing digits of the quotient. Recall that maximum value of the estimation used is $2\beta - \frac{2}{\beta}$ which means that a digit of the quotient must be first produced by another means. In this case y is shifted to the left (*step ten*) so that it has the same number of digits as x. The loop on step eleven will subtract multiples of the shifted copy of y until x is smaller. Since the leading digit of y is greater than or equal to $\beta/2$ this loop will iterate at most two times to produce the desired leading digit of the quotient.

Now the remainder of the digits can be produced. The equation $\hat{q} = \lfloor \frac{x_i\beta + x_{i-1}}{y_t} \rfloor$ is used to fairly accurately approximate the true quotient digit. The estimation can in theory produce an estimation as high as $2\beta - \frac{2}{\beta}$ but by induction the upper quotient digit is correct (*as established on step eleven*) and the estimate must be less than β.

Recall from section 8.1.1 that the estimation is never too low but may be too high. The next step of the estimation process is to refine the estimation. The loop on step 13.5 uses $x_i\beta^2 + x_{i-1}\beta + x_{i-2}$ and $q_{i-t-1}(y_t\beta + y_{t-1})$ as a higher order approximation to adjust the quotient digit.

After both phases of estimation the quotient digit may still be off by a value of one[1]. Steps 13.6 and 13.7 subtract the multiple of the divisor from the dividend (*Similar to step 3.3 of algorithm 8.1* and then subsequently add a multiple of the divisor if the quotient was too large.

Now that the quotient has been determine finializing the result is a matter of clamping the quotient, fixing the sizes and de-normalizing the remainder. An important aspect of this algorithm seemingly overlooked in other descriptions

[1]This is similar to the error introduced by optimizing Barrett reduction.

such as that of Algorithm 14.20 HAC [2, pp. 598] is that when the estimations are being made (*inside the loop on step 13.5*) that the digits y_{t-1}, x_{i-2} and x_{i-1} may lie outside their respective boundaries. For example, if $t = 0$ or $i \leq 1$ then the digits would be undefined. In those cases the digits should respectively be replaced with a zero.

File: bn_mp_div.c

```
001   #ifdef BN_MP_DIV_C
...
014   #ifdef BN_MP_DIV_SMALL
...
```

The implementation of this algorithm differs slightly from the pseudo code presented previously. In this algorithm either of the quotient c or remainder d may be passed as a **NULL** pointer which indicates their value is not desired. For example, the C code to call the division algorithm with only the quotient is

```
mp_div(&a, &b, &c, NULL);  /* c = [a/b] */
```

Lines 110 and 112 handle the two trivial cases of inputs which are division by zero and dividend smaller than the divisor respectively. After the two trivial cases all of the temporary variables are initialized. Line 145 determines the sign of the quotient and line 146 ensures that both x and y are positive.

The number of bits in the leading digit is calculated on line 151. Implictly an mp_int with r digits will require $lg(\beta)(r-1)+k$ bits of precision which when reduced modulo $lg(\beta)$ produces the value of k. In this case k is the number of bits in the leading digit which is exactly what is required. For the algorithm to operate k must equal $lg(\beta) - 1$ and when it does not the inputs must be normalized by shifting them to the left by $lg(\beta) - 1 - k$ bits.

Throughout the variables n and t will represent the highest digit of x and y respectively. These are first used to produce the leading digit of the quotient. The loop beginning on line 182 will produce the remainder of the quotient digits.

The conditional "continue" on line 184 is used to prevent the algorithm from reading past the leading edge of x which can occur when the algorithm eliminates multiple non-zero digits in a single iteration. This ensures that x_i is always non-zero since by definition the digits above the i'th position x must be zero in order for the quotient to be precise[2].

Lines 214, 216 and 222 through 225 manually construct the high accuracy estimations by setting the digits of the two mp_int variables directly.

[2]Precise as far as integer division is concerned.

8.2 Single Digit Helpers

This section briefly describes a series of single digit helper algorithms which come in handy when working with small constants. All of the helper functions assume the single digit input is positive and will treat them as such.

8.2.1 Single Digit Addition and Subtraction

Both addition and subtraction are performed by "cheating" and using mp_set followed by the higher level addition or subtraction algorithms. As a result these algorithms are subtantially simpler with a slight cost in performance.

Algorithm **mp_add_d**.
Input. mp_int a and a mp_digit b
Output. $c = a + b$

1. $t \leftarrow b$ (mp_set)
2. $c \leftarrow a + t$
3. Return(MP_OKAY)

Figure 8.4: Algorithm mp_add_d

Algorithm mp_add_d. This algorithm initiates a temporary mp_int with the value of the single digit and uses algorithm mp_add to add the two values together.

File: bn_mp_add_d.c

```
001   #ifdef BN_MP_ADD_D_C
...
```

Clever use of the letter 't'.

Subtraction

The single digit subtraction algorithm mp_sub_d is essentially the same except it uses mp_sub to subtract the digit from the mp_int.

8.2.2 Single Digit Multiplication

Single digit multiplication arises enough in division and radix conversion that it ought to be implement as a special case of the baseline multiplication algorithm. Essentially this algorithm is a modified version of algorithm s_mp_mul_digs where one of the multiplicands only has one digit.

Algorithm mp_mul_d. This algorithm quickly multiplies an mp_int by a small single digit value. It is specially tailored to the job and has a minimal of overhead. Unlike the full multiplication algorithms this algorithm does not require any significnat temporary storage or memory allocations.

File: bn_mp_mul_d.c

```
001   #ifdef BN_MP_MUL_D_C
...
```

Algorithm **mp_mul_d**.
Input. mp_int a and a mp_digit b
Output. $c = ab$

1. $pa \leftarrow a.used$
2. Grow c to at least $pa + 1$ digits.
3. $oldused \leftarrow c.used$
4. $c.used \leftarrow pa + 1$
5. $c.sign \leftarrow a.sign$
6. $\mu \leftarrow 0$
7. for ix from 0 to $pa - 1$ do
   7.1 $\hat{r} \leftarrow \mu + a_{ix}b$
   7.2 $c_{ix} \leftarrow \hat{r} \pmod{\beta}$
   7.3 $\mu \leftarrow \lfloor \hat{r}/\beta \rfloor$
8. $c_{pa} \leftarrow \mu$
9. for ix from $pa + 1$ to $oldused$ do
   9.1 $c_{ix} \leftarrow 0$
10. Clamp excess digits of c.
11. Return(MP_OKAY).

Figure 8.5: Algorithm mp_mul_d

In this implementation the destination c may point to the same mp_int as the source a since the result is written after the digit is read from the source. This function uses pointer aliases $tmpa$ and $tmpc$ for the digits of a and c respectively.

8.2.3 Single Digit Division

Like the single digit multiplication algorithm, single digit division is also a fairly common algorithm used in radix conversion. Since the divisor is only a single digit a specialized variant of the division algorithm can be used to compute the quotient.

Algorithm **mp_div_d**.
Input. mp_int a and a mp_digit b
Output. $c = \lfloor a/b \rfloor, d = a - cb$

1. If $b = 0$ then return(MP_VAL).
2. If $b = 3$ then use algorithm mp_div_3 instead.
3. Init q to $a.used$ digits.
4. $q.used \leftarrow a.used$
5. $q.sign \leftarrow a.sign$
6. $\hat{w} \leftarrow 0$
7. for ix from $a.used - 1$ down to 0 do
   7.1 $\hat{w} \leftarrow \hat{w}\beta + a_{ix}$
   7.2 If $\hat{w} \geq b$ then
      7.2.1 $t \leftarrow \lfloor \hat{w}/b \rfloor$
      7.2.2 $\hat{w} \leftarrow \hat{w} \pmod b$
   7.3 else
      7.3.1 $t \leftarrow 0$
   7.4 $q_{ix} \leftarrow t$
8. $d \leftarrow \hat{w}$
9. Clamp excess digits of q.
10. $c \leftarrow q$
11. Return(MP_OKAY).

Figure 8.6: Algorithm mp_div_d

Algorithm mp_div_d. This algorithm divides the mp_int a by the single mp_digit b using an optimized approach. Essentially in every iteration of the algorithm another digit of the dividend is reduced and another digit of quotient produced. Provided $b < \beta$ the value of $\hat{w}$ after step 7.1 will be limited such that $0 \leq \lfloor \hat{w}/b \rfloor < \beta$.

If the divisor b is equal to three a variant of this algorithm is used which is called mp_div_3. It replaces the division by three with a multiplication by $\lfloor \beta/3 \rfloor$ and the appropriate shift and residual fixup. In essence it is much like the Barrett reduction from chapter seven.

File: bn_mp_div_d.c
```
001   #ifdef BN_MP_DIV_D_C
...
067   #ifdef BN_MP_DIV_3_C
...
```

Like the implementation of algorithm mp_div this algorithm allows either

of the quotient or remainder to be passed as a **NULL** pointer to indicate the respective value is not required. This allows a trivial single digit modular reduction algorithm, mp_mod_d to be created.

The division and remainder on lines 87 and 88 can be replaced often by a single division on most processors. For example, the 32-bit x86 based processors can divide a 64-bit quantity by a 32-bit quantity and produce the quotient and remainder simultaneously. Unfortunately the GCC compiler does not recognize that optimization and will actually produce two function calls to find the quotient and remainder respectively.

8.2.4 Single Digit Root Extraction

Finding the n'th root of an integer is fairly easy as far as numerical analysis is concerned. Algorithms such as the Newton-Raphson approximation (8.6) series will converge very quickly to a root for any continuous function $f(x)$.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{8.6}$$

In this case the n'th root is desired and $f(x) = x^n - a$ where a is the integer of which the root is desired. The derivative of $f(x)$ is simply $f'(x) = nx^{n-1}$. Of particular importance is that this algorithm will be used over the integers not over the a more continuous domain such as the real numbers. As a result the root found can be above the true root by few and must be manually adjusted. Ideally at the end of the algorithm the n'th root b of an integer a is desired such that $b^n \le a$.

Algorithm **mp_n_root**.
Input. mp_int a and a mp_digit b
Output. $c^b \le a$

1. If b is even and $a.sign = MP_NEG$ return(MP_VAL).
2. $sign \leftarrow a.sign$
3. $a.sign \leftarrow MP_ZPOS$
4. t2 $\leftarrow 2$
5. Loop
   5.1 t1 $\leftarrow$ t2
   5.2 t3 $\leftarrow$ t1^{b-1}
   5.3 t2 $\leftarrow$ t3 $\cdot$ t1
   5.4 t2 $\leftarrow$ t2 $- a$
   5.5 t3 $\leftarrow$ t3 $\cdot b$
   5.6 t3 $\leftarrow \lfloor$t2/t3$\rfloor$
   5.7 t2 $\leftarrow$ t1$-$ t3
   5.8 If t1 $\neq$ t2 then goto step 5.
6. Loop
   6.1 t2 $\leftarrow$ t1^b
   6.2 If t2 $> a$ then
      6.2.1 t1 $\leftarrow$ t1 $- 1$
      6.2.2 Goto step 6.
7. $a.sign \leftarrow sign$
8. $c \leftarrow$ t1
9. $c.sign \leftarrow sign$
10. Return(MP_OKAY).

Figure 8.7: Algorithm mp_n_root

Algorithm mp_n_root. This algorithm finds the integer n'th root of an input using the Newton-Raphson approach. It is partially optimized based on the observation that the numerator of $\frac{f(x)}{f'(x)}$ can be derived from a partial denominator. That is at first the denominator is calculated by finding x^{b-1}. This value can then be multiplied by x and have a subtracted from it to find the numerator. This saves a total of $b-1$ multiplications by t1 inside the loop.

The initial value of the approximation is t2 $= 2$ which allows the algorithm to start with very small values and quickly converge on the root. Ideally this algorithm is meant to find the n'th root of an input where n is bounded by $2 \le n \le 5$.

File: bn_mp_n_root.c

```
001   #ifdef BN_MP_N_ROOT_C
...
```

8.3 Random Number Generation

Random numbers come up in a variety of activities from public key cryptography to simple simulations and various randomized algorithms. Pollard-Rho factoring for example, can make use of random values as starting points to find factors of a composite integer. In this case the algorithm presented is solely for simulations and not intended for cryptographic use.

Algorithm **mp_rand**.
Input. An integer b
Output. A pseudo-random number of b digits

1. $a \leftarrow 0$
2. If $b \leq 0$ return(MP_OKAY)
3. Pick a non-zero random digit d.
4. $a \leftarrow a + d$
5. for ix from 1 to $d - 1$ do
   5.1 $a \leftarrow a \cdot \beta$
   5.2 Pick a random digit d.
   5.3 $a \leftarrow a + d$
6. Return(MP_OKAY).

Figure 8.8: Algorithm mp_rand

Algorithm mp_rand. This algorithm produces a pseudo-random integer of b digits. By ensuring that the first digit is non-zero the algorithm also guarantees that the final result has at least b digits. It relies heavily on a third-part random number generator which should ideally generate uniformly all of the integers from 0 to $\beta - 1$.

File: bn_mp_rand.c

```
001   #ifdef BN_MP_RAND_C
...
```

8.4 Formatted Representations

The ability to emit a radix-n textual representation of an integer is useful for interacting with human parties. For example, the ability to be given a string of characters such as "114585" and turn it into the radix-β equivalent would make it easier to enter numbers into a program.

8.4.1 Reading Radix-n Input

For the purposes of this text we will assume that a simple lower ASCII map (8.9) is used for the values of from 0 to 63 to printable characters. For example, when the character "N" is read it represents the integer 23. The first 16 characters of the map are for the common representations up to hexadecimal. After that they

match the “base64” encoding scheme which are suitable chosen such that they are printable. While outputting as base64 may not be too helpful for human operators it does allow communication via non binary mediums.

Value	Char	Value	Char	Value	Char	Value	Char
0	0	1	1	2	2	3	3
4	4	5	5	6	6	7	7
8	8	9	9	10	A	11	B
12	C	13	D	14	E	15	F
16	G	17	H	18	I	19	J
20	K	21	L	22	M	23	N
24	O	25	P	26	Q	27	R
28	S	29	T	30	U	31	V
32	W	33	X	34	Y	35	Z
36	a	37	b	38	c	39	d
40	e	41	f	42	g	43	h
44	i	45	j	46	k	47	l
48	m	49	n	50	o	51	p
52	q	53	r	54	s	55	t
56	u	57	v	58	w	59	x
60	y	61	z	62	+	63	/

Figure 8.9: Lower ASCII Map

Algorithm **mp_read_radix**.
Input. A string str of length sn and radix r.
Output. The radix-β equivalent mp_int.

1. If $r < 2$ or $r > 64$ return(MP_VAL).
2. $ix \leftarrow 0$
3. If $str_0 =$ "-" then do
   3.1 $ix \leftarrow ix + 1$
   3.2 $sign \leftarrow MP_NEG$
4. else
   4.1 $sign \leftarrow MP_ZPOS$
5. $a \leftarrow 0$
6. for iy from ix to $sn - 1$ do
   6.1 Let y denote the position in the map of str_{iy}.
   6.2 If str_{iy} is not in the map or $y \geq r$ then goto step 7.
   6.3 $a \leftarrow a \cdot r$
   6.4 $a \leftarrow a + y$
7. If $a \neq 0$ then $a.sign \leftarrow sign$
8. Return(MP_OKAY).

Figure 8.10: Algorithm mp_read_radix

Algorithm mp_read_radix. This algorithm will read an ASCII string and produce the radix-β mp_int representation of the same integer. A minus symbol "-" may precede the string to indicate the value is negative, otherwise it is assumed to be positive. The algorithm will read up to sn characters from the input and will stop when it reads a character it cannot map the algorithm stops reading characters from the string. This allows numbers to be embedded as part of larger input without any significant problem.

File: bn_mp_read_radix.c

```
001   #ifdef BN_MP_READ_RADIX_C
...
```

8.4.2 Generating Radix-n Output

Generating radix-n output is fairly trivial with a division and remainder algorithm.

Algorithm **mp_toradix**.
Input. A mp_int a and an integer r
Output. The radix-r representation of a

1. If $r < 2$ or $r > 64$ return(MP_VAL).
2. If $a = 0$ then $str =$ "0" and return(MP_OKAY).
3. $t \leftarrow a$
4. $str \leftarrow$ ""
5. if $t.sign = MP_NEG$ then
   5.1 $str \leftarrow str +$ "-"
   5.2 $t.sign = MP_ZPOS$
6. While ($t \neq 0$) do
   6.1 $d \leftarrow t \pmod r$
   6.2 $t \leftarrow \lfloor t/r \rfloor$
   6.3 Look up d in the map and store the equivalent character in y.
   6.4 $str \leftarrow str + y$
7. If $str_0 =$"$-$" then
   7.1 Reverse the digits $str_1, str_2, \ldots str_n$.
8. Otherwise
   8.1 Reverse the digits $str_0, str_1, \ldots str_n$.
9. Return(MP_OKAY).

Figure 8.11: Algorithm mp_toradix

Algorithm mp_toradix. This algorithm computes the radix-r representation of an mp_int a. The "digits" of the representation are extracted by reducing successive powers of $\lfloor a/r^k \rfloor$ the input modulo r until $r^k > a$. Note that instead of actually dividing by r^k in each iteration the quotient $\lfloor a/r \rfloor$ is saved for the next iteration. As a result a series of trivial $n \times 1$ divisions are required instead of a series of $n \times k$ divisions. One design flaw of this approach is that the digits are produced in the reverse order (see 8.12). To remedy this flaw the digits must be swapped or simply "reversed".

File: bn_mp_toradix.c

```
001   #ifdef BN_MP_TORADIX_C
...
```

Value of a	**Value of** d	**Value of** str
1234	–	–
123	4	“4”
12	3	“43”
1	2	“432”
0	1	“4321”

Figure 8.12: Example of Algorithm mp_toradix.

Chapter 9

Number Theoretic Algorithms

This chapter discusses several fundamental number theoretic algorithms such as the greatest common divisor, least common multiple and Jacobi symbol computation. These algorithms arise as essential components in several key cryptographic algorithms such as the RSA public key algorithm and various Sieve based factoring algorithms.

9.1 Greatest Common Divisor

The greatest common divisor of two integers a and b, often denoted as (a, b) is the largest integer k that is a proper divisor of both a and b. That is, k is the largest integer such that $0 \equiv a \pmod{k}$ and $0 \equiv b \pmod{k}$ occur simultaneously.

The most common approach (cite) is to reduce one input modulo another. That is if a and b are divisible by some integer k and if $qa + r = b$ then r is also divisible by k. The reduction pattern follows $\langle a, b\rangle \rightarrow \langle b, a \bmod b\rangle$.

Algorithm **Greatest Common Divisor (I)**.
Input. Two positive integers a and b greater than zero.
Output. The greatest common divisor (a, b).

1. While $(b > 0)$ do
   1.1 $r \leftarrow a \pmod{b}$
   1.2 $a \leftarrow b$
   1.3 $b \leftarrow r$
2. Return(a).

Figure 9.1: Algorithm Greatest Common Divisor (I)

This algorithm will quickly converge on the greatest common divisor since the residue r tends diminish rapidly. However, divisions are relatively expensive operations to perform and should ideally be avoided. There is another approach based on a similar relationship of greatest common divisors. The faster approach is based on the observation that if k divides both a and b it will also divide $a-b$. In particular, we would like $a - b$ to decrease in magnitude which implies that $b \geq a$.

Algorithm **Greatest Common Divisor (II)**.
Input. Two positive integers a and b greater than zero.
Output. The greatest common divisor (a, b).

1. While $(b > 0)$ do
   1.1 Swap a and b such that a is the smallest of the two.
   1.2 $b \leftarrow b - a$
2. Return(a).

Figure 9.2: Algorithm Greatest Common Divisor (II)

Proof *Algorithm 9.2 will return the greatest common divisor of a and b.* The algorithm in figure 9.2 will eventually terminate since $b \geq a$ the subtraction in step 1.2 will be a value less than b. In other words in every iteration that tuple $\langle a, b\rangle$ decrease in magnitude until eventually $a = b$. Since both a and b are always divisible by the greatest common divisor (*until the last iteration*) and in the last iteration of the algorithm $b = 0$, therefore, in the second to last iteration of the algorithm $b = a$ and clearly $(a, a) = a$ which concludes the proof. **QED**.

As a matter of practicality algorithm 9.1 decreases far too slowly to be useful.

Specially if b is much larger than a such that $b-a$ is still very much larger than a. A simple addition to the algorithm is to divide $b-a$ by a power of some integer p which does not divide the greatest common divisor but will divide $b-a$. In this case $\frac{b-a}{p}$ is also an integer and still divisible by the greatest common divisor.

However, instead of factoring $b-a$ to find a suitable value of p the powers of p can be removed from a and b that are in common first. Then inside the loop whenever $b-a$ is divisible by some power of p it can be safely removed.

Algorithm **Greatest Common Divisor (III)**.
Input. Two positive integers a and b greater than zero.
Output. The greatest common divisor (a, b).

1. $k \leftarrow 0$
2. While a and b are both divisible by p do
   2.1 $a \leftarrow \lfloor a/p \rfloor$
   2.2 $b \leftarrow \lfloor b/p \rfloor$
   2.3 $k \leftarrow k+1$
3. While a is divisible by p do
   3.1 $a \leftarrow \lfloor a/p \rfloor$
4. While b is divisible by p do
   4.1 $b \leftarrow \lfloor b/p \rfloor$
5. While $(b > 0)$ do
   5.1 Swap a and b such that a is the smallest of the two.
   5.2 $b \leftarrow b-a$
   5.3 While b is divisible by p do
      5.3.1 $b \leftarrow \lfloor b/p \rfloor$
6. Return$(a \cdot p^k)$.

Figure 9.3: Algorithm Greatest Common Divisor (III)

This algorithm is based on the first except it removes powers of p first and inside the main loop to ensure the tuple $\langle a, b \rangle$ decreases more rapidly. The first loop on step two removes powers of p that are in common. A count, k, is kept which will present a common divisor of p^k. After step two the remaining common divisor of a and b cannot be divisible by p. This means that p can be safely divided out of the difference $b-a$ so long as the division leaves no remainder.

In particular the value of p should be chosen such that the division on step 5.3.1 occur often. It also helps that division by p be easy to compute. The ideal

choice of p is two since division by two amounts to a right logical shift. Another important observation is that by step five both a and b are odd. Therefore, the diffrence $b - a$ must be even which means that each iteration removes one bit from the largest of the pair.

9.1.1 Complete Greatest Common Divisor

The algorithms presented so far cannot handle inputs which are zero or negative. The following algorithm can handle all input cases properly and will produce the greatest common divisor.

Algorithm **mp_gcd**.
Input. mp_int a and b
Output. The greatest common divisor $c = (a, b)$.

1. If $a = 0$ then
   1.1 $c \leftarrow |b|$
   1.2 Return(MP_OKAY).
2. If $b = 0$ then
   2.1 $c \leftarrow |a|$
   2.2 Return(MP_OKAY).
3. $u \leftarrow |a|, v \leftarrow |b|$
4. $k \leftarrow 0$
5. While $u.used > 0$ and $v.used > 0$ and $u_0 \equiv v_0 \equiv 0 \pmod 2$
   5.1 $k \leftarrow k + 1$
   5.2 $u \leftarrow \lfloor u/2 \rfloor$
   5.3 $v \leftarrow \lfloor v/2 \rfloor$
6. While $u.used > 0$ and $u_0 \equiv 0 \pmod 2$
   6.1 $u \leftarrow \lfloor u/2 \rfloor$
7. While $v.used > 0$ and $v_0 \equiv 0 \pmod 2$
   7.1 $v \leftarrow \lfloor v/2 \rfloor$
8. While $v.used > 0$
   8.1 If $|u| > |v|$ then
      8.1.1 Swap u and v.
   8.2 $v \leftarrow |v| - |u|$
   8.3 While $v.used > 0$ and $v_0 \equiv 0 \pmod 2$
      8.3.1 $v \leftarrow \lfloor v/2 \rfloor$
9. $c \leftarrow u \cdot 2^k$
10. Return(MP_OKAY).

Figure 9.4: Algorithm mp_gcd

Algorithm mp_gcd. This algorithm will produce the greatest common divisor of two mp_ints a and b. The algorithm was originally based on Algorithm B of Knuth [1, pp. 338] but has been modified to be simpler to explain. In theory it achieves the same asymptotic working time as Algorithm B and in practice this appears to be true.

The first two steps handle the cases where either one of or both inputs are zero. If either input is zero the greatest common divisor is the largest input or zero if they are both zero. If the inputs are not trivial than u and v are assigned the absolute values of a and b respectively and the algorithm will proceed to

reduce the pair.

Step five will divide out any common factors of two and keep track of the count in the variable k. After this step, two is no longer a factor of the remaining greatest common divisor between u and v and can be safely evenly divided out of either whenever they are even. Step six and seven ensure that the u and v respectively have no more factors of two. At most only one of the while–loops will iterate since they cannot both be even.

By step eight both of u and v are odd which is required for the inner logic. First the pair are swapped such that v is equal to or greater than u. This ensures that the subtraction on step 8.2 will always produce a positive and even result. Step 8.3 removes any factors of two from the difference u to ensure that in the next iteration of the loop both are once again odd.

After $v = 0$ occurs the variable u has the greatest common divisor of the pair $\langle u, v \rangle$ just after step six. The result must be adjusted by multiplying by the common factors of two (2^k) removed earlier.

File: bn_mp_gcd.c

```
001   #ifdef BN_MP_GCD_C
...
```

This function makes use of the macros mp_iszero and mp_iseven. The former evaluates to 1 if the input mp_int is equivalent to the integer zero otherwise it evaluates to 0. The latter evaluates to 1 if the input mp_int represents a non-zero even integer otherwise it evaluates to 0. Note that just because mp_iseven may evaluate to 0 does not mean the input is odd, it could also be zero. The three trivial cases of inputs are handled on lines 22 through 27. After those lines the inputs are assumed to be non-zero.

Lines 34 and 34 make local copies u and v of the inputs a and b respectively. At this point the common factors of two must be divided out of the two inputs. The block starting at line 41 removes common factors of two by first counting the number of trailing zero bits in both. The local integer k is used to keep track of how many factors of 2 are pulled out of both values. It is assumed that the number of factors will not exceed the maximum value of a C "int" data type[1].

At this point there are no more common factors of two in the two values. The divisions by a power of two on lines 59 and 65 remove any independent factors of two such that both u and v are guaranteed to be an odd integer before hitting the main body of the algorithm. The while loop on line 70 performs the reduction

[1]Strictly speaking no array in C may have more than entries than are accessible by an "int" so this is not a limitation.

of the pair until v is equal to zero. The unsigned comparison and subtraction algorithms are used in place of the full signed routines since both values are guaranteed to be positive and the result of the subtraction is guaranteed to be non-negative.

9.2 Least Common Multiple

The least common multiple of a pair of integers is their product divided by their greatest common divisor. For two integers a and b the least common multiple is normally denoted as $[a, b]$ and numerically equivalent to $\frac{ab}{(a,b)}$. For example, if $a = 2 \cdot 2 \cdot 3 = 12$ and $b = 2 \cdot 3 \cdot 3 \cdot 7 = 126$ the least common multiple is $\frac{126}{(12,126)} = \frac{126}{6} = 21$.

The least common multiple arises often in coding theory as well as number theory. If two functions have periods of a and b respectively they will collide, that is be in synchronous states, after only $[a, b]$ iterations. This is why, for example, random number generators based on Linear Feedback Shift Registers (LFSR) tend to use registers with periods which are co-prime (*e.g. the greatest common divisor is one.*). Similarly in number theory if a composite n has two prime factors p and q then maximal order of any unit of $\mathbb{Z}/n\mathbb{Z}$ will be $[p-1, q-1]$.

Algorithm **mp_lcm**.
Input. mp_int a and b
Output. The least common multiple $c = [a, b]$.

1. $c \leftarrow (a, b)$
2. $t \leftarrow a \cdot b$
3. $c \leftarrow \lfloor t/c \rfloor$
4. Return(MP_OKAY).

Figure 9.5: Algorithm mp_lcm

Algorithm mp_lcm. This algorithm computes the least common multiple of two mp_int inputs a and b. It computes the least common multiple directly by dividing the product of the two inputs by their greatest common divisor.

File: bn_mp_lcm.c

```
001   #ifdef BN_MP_LCM_C
...
```

9.3 Jacobi Symbol Computation

To explain the Jacobi Symbol we shall first discuss the Legendre function[2] off which the Jacobi symbol is defined. The Legendre function computes whether or not an integer a is a quadratic residue modulo an odd prime p. Numerically it is equivalent to equation 9.1.

– *Tom, don't be an ass, cite your source here...!*

$$a^{(p-1)/2} \equiv \begin{array}{rl} -1 & \text{if } a \text{ is a quadratic non-residue.} \\ 0 & \text{if } a \text{ divides } p. \\ 1 & \text{if } a \text{ is a quadratic residue.} \end{array} \pmod{p} \tag{9.1}$$

Proof. *Equation 9.1 correctly identifies the residue status of an integer a modulo a prime p.* An integer a is a quadratic residue if the following equation has a solution.

$$x^2 \equiv a \pmod{p} \tag{9.2}$$

Consider the following equation.

$$0 \equiv x^{p-1} - 1 \equiv \left\{ \left(x^2\right)^{(p-1)/2} - a^{(p-1)/2} \right\} + \left(a^{(p-1)/2} - 1 \right) \pmod{p} \tag{9.3}$$

Whether equation 9.2 has a solution or not equation 9.3 is always true. If $a^{(p-1)/2} - 1 \equiv 0 \pmod{p}$ then the quantity in the braces must be zero. By reduction,

$$\begin{aligned} \left(x^2\right)^{(p-1)/2} - a^{(p-1)/2} &\equiv 0 \\ \left(x^2\right)^{(p-1)/2} &\equiv a^{(p-1)/2} \\ x^2 &\equiv a \pmod{p} \end{aligned} \tag{9.4}$$

As a result there must be a solution to the quadratic equation and in turn a must be a quadratic residue. If a does not divide p and a is not a quadratic residue then the only other value $a^{(p-1)/2}$ may be congruent to is -1 since

$$0 \equiv a^{p-1} - 1 \equiv (a^{(p-1)/2} + 1)(a^{(p-1)/2} - 1) \pmod{p} \tag{9.5}$$

One of the terms on the right hand side must be zero. **QED**

[2] Arrg. What is the name of this?

9.3.1 Jacobi Symbol

The Jacobi symbol is a generalization of the Legendre function for any odd non prime moduli p greater than 2. If $p = \prod_{i=0}^{n} p_i$ then the Jacobi symbol $\left(\frac{a}{p}\right)$ is equal to the following equation.

$$\left(\frac{a}{p}\right) = \left(\frac{a}{p_0}\right)\left(\frac{a}{p_1}\right)\cdots\left(\frac{a}{p_n}\right) \tag{9.6}$$

By inspection if p is prime the Jacobi symbol is equivalent to the Legendre function. The following facts[3] will be used to derive an efficient Jacobi symbol algorithm. Where p is an odd integer greater than two and $a, b \in \mathbb{Z}$ the following are true.

1. $\left(\frac{a}{p}\right)$ equals -1, 0 or 1.
2. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.
3. If $a \equiv b$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.
4. $\left(\frac{2}{p}\right)$ equals 1 if $p \equiv 1$ or 7 (mod 8). Otherwise, it equals -1.
5. $\left(\frac{a}{p}\right) \equiv \left(\frac{p}{a}\right) \cdot (-1)^{(p-1)(a-1)/4}$. More specifically $\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right)$ if $p \equiv a \equiv 1 \pmod 4$.

Using these facts if $a = 2^k \cdot a'$ then

$$\begin{aligned}\left(\frac{a}{p}\right) &= \left(\frac{2^k}{p}\right)\left(\frac{a'}{p}\right) \\ &= \left(\frac{2}{p}\right)^k \left(\frac{a'}{p}\right)\end{aligned} \tag{9.7}$$

By fact five,

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right) \cdot (-1)^{(p-1)(a-1)/4} \tag{9.8}$$

[3] See HAC [2, pp. 72-74] for further details.

Subsequently by fact three since $p \equiv (p \bmod a) \pmod a$ then

$$\left(\frac{a}{p}\right) = \left(\frac{p \bmod a}{a}\right) \cdot (-1)^{(p-1)(a-1)/4} \tag{9.9}$$

By putting both observations into equation 9.7 the following simplified equation is formed.

$$\left(\frac{a}{p}\right) = \left(\frac{2}{p}\right)^k \left(\frac{p \bmod a'}{a'}\right) \cdot (-1)^{(p-1)(a'-1)/4} \tag{9.10}$$

The value of $\left(\frac{p \bmod a'}{a'}\right)$ can be found by using the same equation recursively. The value of $\left(\frac{2}{p}\right)^k$ equals 1 if k is even otherwise it equals $\left(\frac{2}{p}\right)$. Using this approach the factors of p do not have to be known. Furthermore, if $(a, p) = 1$ then the algorithm will terminate when the recursion requests the Jacobi symbol computation of $\left(\frac{1}{a'}\right)$ which is simply 1.

Algorithm **mp_jacobi**.
Input. mp_int a and p, $a \geq 0$, $p \geq 3$, $p \equiv 1 \pmod 2$
Output. The Jacobi symbol $c = \left(\frac{a}{p}\right)$.

1. If $a = 0$ then
   1.1 $c \leftarrow 0$
   1.2 Return(MP_OKAY).
2. If $a = 1$ then
   2.1 $c \leftarrow 1$
   2.2 Return(MP_OKAY).
3. $a' \leftarrow a$
4. $k \leftarrow 0$
5. While $a'.used > 0$ and $a'_0 \equiv 0 \pmod 2$
   5.1 $k \leftarrow k + 1$
   5.2 $a' \leftarrow \lfloor a'/2 \rfloor$
6. If $k \equiv 0 \pmod 2$ then
   6.1 $s \leftarrow 1$
7. else
   7.1 $r \leftarrow p_0 \pmod 8$
   7.2 If $r = 1$ or $r = 7$ then
      7.2.1 $s \leftarrow 1$
   7.3 else
      7.3.1 $s \leftarrow -1$
8. If $p_0 \equiv a'_0 \equiv 3 \pmod 4$ then
   8.1 $s \leftarrow -s$
9. If $a' \neq 1$ then
   9.1 $p' \leftarrow p \pmod{a'}$
   9.2 $s \leftarrow s \cdot \text{mp_jacobi}(p', a')$
10. $c \leftarrow s$
11. Return(MP_OKAY).

Figure 9.6: Algorithm mp_jacobi

Algorithm mp_jacobi. This algorithm computes the Jacobi symbol for an arbitrary positive integer a with respect to an odd integer p greater than three. The algorithm is based on algorithm 2.149 of HAC [2, pp. 73].

Step numbers one and two handle the trivial cases of $a = 0$ and $a = 1$ respectively. Step five determines the number of two factors in the input a. If k is even than the term $\left(\frac{2}{p}\right)^k$ must always evaluate to one. If k is odd than the term evaluates to one if p_0 is congruent to one or seven modulo eight, otherwise

it evaluates to -1. After the the $\left(\frac{2}{p}\right)^k$ term is handled the $(-1)^{(p-1)(a'-1)/4}$ is computed and multiplied against the current product s. The latter term evaluates to one if both p and a' are congruent to one modulo four, otherwise it evaluates to negative one.

By step nine if a' does not equal one a recursion is required. Step 9.1 computes $p' \equiv p \pmod{a'}$ and will recurse to compute $\left(\frac{p'}{a'}\right)$ which is multiplied against the current Jacobi product.

File: bn_mp_jacobi.c

```
001   #ifdef BN_MP_JACOBI_C
...
```

As a matter of practicality the variable a' as per the pseudo-code is reprensented by the variable $a1$ since the $'$ symbol is not valid for a C variable name character.

The two simple cases of $a = 0$ and $a = 1$ are handled at the very beginning to simplify the algorithm. If the input is non-trivial the algorithm has to proceed compute the Jacobi. The variable s is used to hold the current Jacobi product. Note that s is merely a C "int" data type since the values it may obtain are merely -1, 0 and 1.

After a local copy of a is made all of the factors of two are divided out and the total stored in k. Technically only the least significant bit of k is required, however, it makes the algorithm simpler to follow to perform an addition. In practice an exclusive-or and addition have the same processor requirements and neither is faster than the other.

Line 40 through 68 determines the value of $\left(\frac{2}{p}\right)^k$. If the least significant bit of k is zero than k is even and the value is one. Otherwise, the value of s depends on which residue class p belongs to modulo eight. The value of $(-1)^{(p-1)(a'-1)/4}$ is compute and multiplied against s on lines 72 through 75.

Finally, if $a1$ does not equal one the algorithm must recurse and compute $\left(\frac{p'}{a'}\right)$.

– *Comment about default s and such...*

9.4 Modular Inverse

The modular inverse of a number actually refers to the modular multiplicative inverse. Essentially for any integer a such that $(a, p) = 1$ there exist another integer b such that $ab \equiv 1 \pmod{p}$. The integer b is called the multiplicative

inverse of a which is denoted as $b = a^{-1}$. Technically speaking modular inversion is a well defined operation for any finite ring or field not just for rings and fields of integers. However, the former will be the matter of discussion.

The simplest approach is to compute the algebraic inverse of the input. That is to compute $b \equiv a^{\Phi(p)-1}$. If $\Phi(p)$ is the order of the multiplicative subgroup modulo p then b must be the multiplicative inverse of a. The proof of which is trivial.

$$ab \equiv a\left(a^{\Phi(p)-1}\right) \equiv a^{\Phi(p)} \equiv a^0 \equiv 1 \pmod{p} \tag{9.11}$$

However, as simple as this approach may be it has two serious flaws. It requires that the value of $\Phi(p)$ be known which if p is composite requires all of the prime factors. This approach also is very slow as the size of p grows.

A simpler approach is based on the observation that solving for the multiplicative inverse is equivalent to solving the linear Diophantine[4] equation.

$$ab + pq = 1 \tag{9.12}$$

Where a, b, p and q are all integers. If such a pair of integers $\langle b, q\rangle$ exist than b is the multiplicative inverse of a modulo p. The extended Euclidean algorithm (Knuth [1, pp. 342]) can be used to solve such equations provided $(a, p) = 1$. However, instead of using that algorithm directly a variant known as the binary Extended Euclidean algorithm will be used in its place. The binary approach is very similar to the binary greatest common divisor algorithm except it will produce a full solution to the Diophantine equation.

9.4.1 General Case

[4] See LeVeque [20, pp. 40-43] for more information.

Algorithm **mp_invmod**.
Input. mp_int a and b, $(a, b) = 1$, $p \geq 2$, $0 < a < p$.
Output. The modular inverse $c \equiv a^{-1} \pmod{b}$.

1. If $b \leq 0$ then return(MP_VAL).
2. If $b_0 \equiv 1 \pmod 2$ then use algorithm fast_mp_invmod.
3. $x \leftarrow |a|, y \leftarrow b$
4. If $x_0 \equiv y_0 \equiv 0 \pmod 2$ then return(MP_VAL).
5. $B \leftarrow 0, C \leftarrow 0, A \leftarrow 1, D \leftarrow 1$
6. While $u.used > 0$ and $u_0 \equiv 0 \pmod 2$
   6.1 $u \leftarrow \lfloor u/2 \rfloor$
   6.2 If ($A.used > 0$ and $A_0 \equiv 1 \pmod 2$) or ($B.used > 0$ and $B_0 \equiv 1 \pmod 2$) then
      6.2.1 $A \leftarrow A + y$
      6.2.2 $B \leftarrow B - x$
   6.3 $A \leftarrow \lfloor A/2 \rfloor$
   6.4 $B \leftarrow \lfloor B/2 \rfloor$
7. While $v.used > 0$ and $v_0 \equiv 0 \pmod 2$
   7.1 $v \leftarrow \lfloor v/2 \rfloor$
   7.2 If ($C.used > 0$ and $C_0 \equiv 1 \pmod 2$) or ($D.used > 0$ and $D_0 \equiv 1 \pmod 2$) then
      7.2.1 $C \leftarrow C + y$
      7.2.2 $D \leftarrow D - x$
   7.3 $C \leftarrow \lfloor C/2 \rfloor$
   7.4 $D \leftarrow \lfloor D/2 \rfloor$
8. If $u \geq v$ then
   8.1 $u \leftarrow u - v$
   8.2 $A \leftarrow A - C$
   8.3 $B \leftarrow B - D$
9. else
   9.1 $v \leftarrow v - u$
   9.2 $C \leftarrow C - A$
   9.3 $D \leftarrow D - B$
10. If $u \neq 0$ goto step 6.
11. If $v \neq 1$ return(MP_VAL).
12. While $C \leq 0$ do
   12.1 $C \leftarrow C + b$
13. While $C \geq b$ do
   13.1 $C \leftarrow C - b$
14. $c \leftarrow C$
15. Return(MP_OKAY).

Algorithm mp_invmod. This algorithm computes the modular multi-

plicative inverse of an integer a modulo an integer b. This algorithm is a variation of the extended binary Euclidean algorithm from HAC [2, pp. 608]. It has been modified to only compute the modular inverse and not a complete Diophantine solution.

If $b \le 0$ than the modulus is invalid and MP_VAL is returned. Similarly if both a and b are even then there cannot be a multiplicative inverse for a and the error is reported.

The astute reader will observe that steps seven through nine are very similar to the binary greatest common divisor algorithm mp_gcd. In this case the other variables to the Diophantine equation are solved. The algorithm terminates when $u = 0$ in which case the solution is

$$Ca + Db = v \tag{9.13}$$

If v, the greatest common divisor of a and b is not equal to one then the algorithm will report an error as no inverse exists. Otherwise, C is the modular inverse of a. The actual value of C is congruent to, but not necessarily equal to, the ideal modular inverse which should lie within $1 \le a^{-1} < b$. Step numbers twelve and thirteen adjust the inverse until it is in range. If the original input a is within $0 < a < p$ then only a couple of additions or subtractions will be required to adjust the inverse.

File: bn_mp_invmod.c

```
001   #ifdef BN_MP_INVMOD_C
...
022   #ifdef BN_FAST_MP_INVMOD_C
...
029   #ifdef BN_MP_INVMOD_SLOW_C
...
```

Odd Moduli

When the modulus b is odd the variables A and C are fixed and are not required to compute the inverse. In particular by attempting to solve the Diophantine $Cb + Da = 1$ only B and D are required to find the inverse of a.

The algorithm fast_mp_invmod is a direct adaptation of algorithm mp_invmod with all all steps involving either A or C removed. This optimization will halve the time required to compute the modular inverse.

9.5 Primality Tests

A non-zero integer a is said to be prime if it is not divisible by any other integer excluding one and itself. For example, $a = 7$ is prime since the integers $2 \ldots 6$ do not evenly divide a. By contrast, $a = 6$ is not prime since $a = 6 = 2 \cdot 3$.

Prime numbers arise in cryptography considerably as they allow finite fields to be formed. The ability to determine whether an integer is prime or not quickly has been a viable subject in cryptography and number theory for considerable time. The algorithms that will be presented are all probablistic algorithms in that when they report an integer is composite it must be composite. However, when the algorithms report an integer is prime the algorithm may be incorrect.

As will be discussed it is possible to limit the probability of error so well that for practical purposes the probablity of error might as well be zero. For the purposes of these discussions let n represent the candidate integer of which the primality is in question.

9.5.1 Trial Division

Trial division means to attempt to evenly divide a candidate integer by small prime integers. If the candidate can be evenly divided it obviously cannot be prime. By dividing by all primes $1 < p \le \sqrt{n}$ this test can actually prove whether an integer is prime. However, such a test would require a prohibitive amount of time as n grows.

Instead of dividing by every prime, a smaller, more mangeable set of primes may be used instead. By performing trial division with only a subset of the primes less than $\sqrt{n} + 1$ the algorithm cannot prove if a candidate is prime. However, often it can prove a candidate is not prime.

The benefit of this test is that trial division by small values is fairly efficient. Specially compared to the other algorithms that will be discussed shortly. The probability that this approach correctly identifies a composite candidate when tested with all primes upto q is given by $1 - \frac{1.12}{ln(q)}$. The graph (9.7) demonstrates the probability of success for the range $3 \le q \le 100$.

At approximately $q = 30$ the gain of performing further tests diminishes fairly quickly. At $q = 90$ further testing is generally not going to be of any practical use. In the case of LibTomMath the default limit $q = 256$ was chosen since it is not too high and will eliminate approximately 80% of all candidate